On Generalized Weights for Codes over Finite Rings

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Abstract

We generalize the definition for higher weights for codes over rings and define weight enumerators corresponding to these weights. We give bounds for these weights and provide MacWilliams relations for the weight enumerators. We determine these weights for some codes over $\mathbb{Z}_4$.

Key Words: Codes over Rings, Higher Weights, Generalized Lee Weights, Weight Enumerators, Singleton Bound.

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1 Introduction

For a linear code over a finite field, Helleseth, Kløve and Mykkeltveit [15] introduced generalized Hamming weights while studying the weight distribution of irreducible cyclic codes and later Wei ([27]) rediscovered the idea of generalized Hamming weights. Following these, numerous papers dealing with these weights have been published (cf. [13], [26] etc.). Recently, the generalized Hamming weights for codes over $\mathbb{Z}_4$ have been defined and studied, see [1], [28], [29], [4] and [16] for example. In this work we generalize the definition to rings giving special attention to the ring $\mathbb{Z}_4$.

Let R be a finite commutative ring, (we assume that a ring has a multiplicative identity). Let $^- : R \to R$ be an involution, we will allow $^-$ to be the identity. A *code* of length n over a ring R is a subset of the free module R^n of rank n and the code is *linear* if it is an R-submodule of R^n.

We define the *inner product* over R by

$$[v, w] = \sum v_i \overline{w_i}. \tag{1}$$

Then $C^\perp = \{w \mid [w, v] = 0 \text{ for all } v \in R^n\}$. We use an involution because it is required for some rings when building complex unimodular lattices from self-dual codes. For other rings, the involution is simply taken to be the identity.

For a linear code C of length n over R, we define the *rank* of C, denoted by rank(C), to be the minimum number of generators of C and define the *free rank* of C, denoted by f-rank(C), to be the maximum of the ranks of R-free submodules of C. We shall say that a code is free if the free rank is equal to the rank, that is, a code is a free R-submodule.

Define the following norm for a vector $\boldsymbol{v} \in R^n$:

$$||\boldsymbol{v}|| = |\text{supp}(\boldsymbol{v})| \tag{2}$$

where

$$\text{supp}(\boldsymbol{v}) = \{i \mid v_i \neq 0\} \tag{3}$$

We extend this norm to subcodes, specifically let C be a code of length n and let D be any subcode of C. Define

$$||D|| = |\text{supp}(D)| \tag{4}$$

where

$$\begin{aligned} \text{supp}(D) &= \{i \mid \text{there exists } \boldsymbol{v} \in D \text{ with } \ v_i \neq 0\} \\ &= \bigcup_{\boldsymbol{v} \in D} \text{supp}(\mathbf{v}). \end{aligned}$$

For a linear code C over a ring R and any g, $1 \leq g \leq \text{rank}(C)$, we define the g-th *generalized Hamming weight with respect to rank* (GHWR) as follows:

$$d_g^H(C) := \min\{||D|| : \ D \text{ is a } R\text{-submodule of } C \text{ with } \text{rank}(D) = r\}. \tag{5}$$

We note that the minimum Hamming weight of a code C is $d_1^H(C)$. In [18], they introduced the GHWR of a linear code C over a finite chain ring and studied some properties of the GHWR.

For any g, $1 \leq g \leq \text{rank}(C)$, we define the higher weight spectrum as

$$A_i^g = |\{D \subseteq C \mid \text{rank}(D) = g, \ ||D|| = i\}| \tag{6}$$

which naturally gives higher weight enumerators

$$W_C^g(y) = \sum A_i^g x^{n-i} y^i. \tag{7}$$

These definitions are of course, the natural extensions of the definitions used for codes over finite fields. The next two extensions are a broader generalization of these ideas.

Let R be a ring and let $a \sim b$ if $a = bu$ for u a unit in R. Let $[a_1], [a_2], \ldots, [a_s]$ denote the non-zero equivalence classes under this relation.

Any linear code over R has a generator matrix which can be put in the following form:

$$\begin{pmatrix} a_1 I_{k_1} & A_{1,2} & A_{1,3} & A_{1,4} & \cdots & \cdots & A_{1,s+1} \\ 0 & a_2 I_{k_2} & a_2 A_{2,3} & a_2 A_{2,4} & \cdots & \cdots & a_2 A_{2,s+1} \\ 0 & 0 & a_3 I_{k_3} & a_3 A_{3,4} & \cdots & \cdots & a_3 A_{3,s+1} \\ \vdots & \vdots & 0 & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_s I_{k_s} & a_s A_{s,s+1} \end{pmatrix}, \tag{8}$$

where $A_{i,j}$ are binary matrices for $i > 1$. A code of this form is said to be of *type* $\{k_1, k_2, k_3, \ldots, k_s\}$ and has $\prod_{i=1}^{s} |a_i R|^{k_i}$ elements, where $a_i R = \{x \mid x = a_i r \text{ for some } r \in R\}$.

For a linear code C over R define

$$\delta_{k_1,\ldots,k_s}(C) = \min\left\{||D|| \mid D \subseteq C, \ \text{type}(D) = \{k_1, \ldots, k_s\}\right\}. \tag{9}$$

We extend the definition of the higher weight spectrum as

$$A_i^{k_1,k_2,\ldots,k_s} = |\{D \subseteq C \mid \text{type}(D) = \{k_1, \ldots, k_s\}, \ ||D|| = i\}| \tag{10}$$

which naturally extends higher weight enumerators as follows:

$$W_C^{k_1,\ldots,k_s}(y) = \sum A_i^{k_1,\ldots,k_s} x^{n-i} y^i. \tag{11}$$

Hence for each type we have a weight enumerator.

Theorem 1.1 *Let C be a code over R then*

$$\begin{aligned} W_C(x,y) &= W^{0,0,\ldots,0}(x,y) + |[a_1]| W^{0,1,0,\ldots,0}(x,y) + |[a_2]| W^{0,0,1,0,\ldots,0}(x,y) + \ldots \\ &\quad + |[a_s]| W^{0,0,\ldots,0,1}(x,y) \\ &= W^{0,0,\ldots,0}(x,y) + \sum_{i=1}^{s} |a_i] W^{\delta_{i1},\delta_{i2},\ldots,\delta_{is}}(x,y) \end{aligned}$$

where $W_C(x,y)$ is the Hamming weight enumerator.

Proof. Any vector generates a subcode of rank 1, of one of the types given above. Each rank 1 subcode is generated by any of its $|[a_i]|$ vectors if it is type $\{\delta_{i1}, \delta_{i2}, \ldots, \delta_{is}\}$. $\square$

If C is a code over $\mathbb{F}_2 + u\mathbb{F}_2$ or $\mathbb{Z}_4$ then the image under the corresponding Gray map of a linear subcode D has support $2||D||$, since any non-zero coordinate is mapped to two non-zero coordinates. Of course, it is necessary for the subcode to be linear for this to be true. If the ring is $\mathbb{F}_2 + u\mathbb{F}_2$ then the image is linear, but in neither case would it account for all binary subcodes. For example, the image of the ambient space of length 1 over $\mathbb{F}_2 + u\mathbb{F}_2$ is $\mathbb{F}_2^2$, but the subcode $\{00, 10\}$ is a binary subcode but corresponds to a non-linear subcode of $\mathbb{F}_2 + u\mathbb{F}_2$.

1.1 Codes over $\mathbb{Z}_4$

Additionally, we give an alternate definition for the higher weight of a code over $\mathbb{Z}_4$ has also been given in [1], [28], [29], [4], and [16]. In [19], Hove studied the concept of generalized Lee weights for codes over $\mathbb{Z}_4$ with respect to the order of a code. Our definition of generalized Lee weights is another natural extension of generalized Hamming weights.

It is known that a linear code C of length n over $\mathbb{Z}_4$ is permutation-equivalent to a linear code with generator matrix of the form

$$\begin{pmatrix} I_{k_1} & X & Y \\ 0 & 2I_{k_2} & 2Z \end{pmatrix}, \tag{12}$$

where X and Z are binary matrices and Y is a matrix over $\mathbb{Z}_4$. In this case, it gives that $|C| = 4^{k_1}2^{k_2}$ and $\text{rank}(C) = k_1 + k_2$. We shall define a code with a generator matrix of the form given in matrix (12) as being of type $\{k_1, k_2\}$ and then say C is an $[n; k_1, k_2]$ code. Sometimes we also write (12) as

$$G = \begin{bmatrix} G_1 \\ 2G_2 \end{bmatrix},$$

where G_1 and G_2 are $k_1 \times n$ and $k_2 \times n$ matrices over $\mathbb{Z}_4$. Let $\hat{C}$ denote the subcode $[n; 0, k_1]$ of C generated by the matrix $[2G_1]$ and let $\tilde{C}$ denote the subcode $[n; 0, k_1 + k_2]$ of C with generator matrix $\begin{bmatrix} 2G_1 \\ 2G_2 \end{bmatrix}$ (see [1]). We will need these in Theorem 2.4.

A vector $\boldsymbol{v}$ is a 2-*linear combination* of the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ if $\boldsymbol{v} = \lambda_1\boldsymbol{v}_1 + \ldots + \lambda_k\boldsymbol{v}_k$ with $\lambda_i \in \mathbb{Z}_2$ for $1 \leq i \leq k$. A subset $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_k\}$ of C is called a 2-*basis* for C if for each $i = 1, 2, ..., k-1$, $2\boldsymbol{v}_i$ is a $2-$linear combination of $\boldsymbol{v}_{i+1}, ..., \boldsymbol{v}_k$, $2\boldsymbol{v}_k = 0$, C is the 2-linear span of S and S is 2-linearly independent [4]. The number of elements in a 2-basis for C is called the 2-*dimension* of C and is denoted by 2-dim(C). It is easy to verify that the rows of the matrix

$$\begin{pmatrix} I_{k_1} & X & Y \\ 2I_{k_1} & 2X & 2Y \\ 0 & 2I_{k_2} & 2Z \end{pmatrix} \tag{13}$$

form a 2-basis for the code C generated by matrix (12). Thus the 2-dimension of C is $2k_1 + k_2$.

For a vector $\boldsymbol{x} \in \mathbb{Z}_4^n$, we denote the Hamming weight and Lee weight by $\mathrm{wt}(\boldsymbol{x})$ and L-$\mathrm{wt}(\boldsymbol{x})$, respectively.

Let C be a linear code of length n over $\mathbb{Z}_4$. Let $A(C)$ be the $|C| \times n$ array of all codewords in C. It is well-known that each column of $A(C)$ corresponds to the following three cases: (i) the column contains only 0 (ii) the column contains 0 and 2 equally often (iii) the column contains all elements of $\mathbb{Z}_4$ equally often (cf. [29]). For the three columns (i), (ii) and (iii), we define the *Lee support weights* of these columns by 0, 2 and 1 respectively. Thus we define the *Lee support weight* $\mathrm{wt}_L(C)$ of C by the sum of the Lee support weights of all columns of $A(C)$. For example, if

$$C = \{(0,0,0), (1,0,1), (2,0,2), (3,0,3), (0,2,2), (1,2,3), (2,2,0), (3,2,1)\},$$

then $\mathrm{wt}_L(C) = 1 + 2 + 1 = 4$. We remark that if C is generated by only one vector $\boldsymbol{x}$, then the Lee support weight $\mathrm{wt}_L(C)$ corresponds to the original Lee weight L-$\mathrm{wt}(\boldsymbol{x})$ of $\boldsymbol{x}$. Now, for $1 \le r \le \mathrm{rank}(C)$, we define the r-th *generalized Lee weight with respect to rank* (GLWR) $d_r^L(C)$ of C as follows:

$$d_r^L(C) := \min\{\mathrm{wt}_L(D) : \ D \text{ is a } \mathbb{Z}_4\text{-submodule of } C \text{ with } \mathrm{rank}(D) = r\}.$$

We note that $d_1^L(C)$ corresponds to the minimum Lee weight of C. And as a connection between the GHWR and the GLWR for a linear code C over $\mathbb{Z}_4$, we remark that

$$d_r^L(C) \le 2d_r^H(C). \tag{14}$$

Additionally, we define the (k_1, k_2)-*generalized Lee weight with respect to type* as follows:

$$d_{k_1,k_2}^L := \min\{\mathrm{wt}_L(D) : \ D \text{ is a } \mathbb{Z}_4\text{-submodule of } C \text{ with type } \{k_1, k_2\}\ \}.$$

Also, for $1 \le r \le 2k_1 + k_2$, we define the r-th *generalized Lee weight with respect to 2-dimension* (GLWT) of C as follows:

$$2\text{-}d_r^L(C) := \min\{\mathrm{wt}_L(D) : \ D \text{ is a } \mathbb{Z}_4\text{-submodule of } C \text{ with } 2\text{-}\dim(D) = r\}.$$

Note that with respect to 2-dimension 2-$d_1^L(C)$ does not always corresponds to the minimum Lee weight of C.
In each case, the set $\{d_r^L(C)\}$ or $\{d_{k_1,k_2}^L(C)\}$ or $\{2\text{-}d_r^L(C)\}$ is called the *Lee weight hierarchy* of C.

In this paper, we shall derive several basic properties of these weights.

2 Subcodes

It is well known how many subspaces of dimension h there are of a space of dimension r over $\mathbb{F}_q$, i.e. $\begin{bmatrix} r \\ h \end{bmatrix}_q$, where $\begin{bmatrix} r \\ h \end{bmatrix}_q = \frac{(q^r-1)(q^r-q)\dots(q^r-q^{h-1})}{(q^h-1)(q^h-q)\dots(q^h-q^{h-1})}$.

We shall generalize this for codes over $\mathbb{Z}_4$. The situation here is more complex because the subcodes can have a variety of different types. In fact there is no formula to determine the number of subcodes of rank h in a code of rank r over $\mathbb{Z}_4$. For example, consider the trivial case of the space $\mathbb{Z}_4$, which is a code of rank 1 over $\mathbb{Z}_4$ which has subcodes of rank 1, $\{0, 2\}$, and $\mathbb{Z}_4$. The code $\{0, 2\}$ is a code of rank 1 and has only itself as a subcode of rank 1. Hence no formula determines the number of subcodes for a specific rank but we shall find a formula based on type.

Let C be a free code over $\mathbb{Z}_4$ of rank r, i.e. its rank and free rank are identical. We shall count the number of free subcodes of C of rank h and denote it by $\begin{bmatrix} r \\ h \end{bmatrix}_4$.

A vector generates a free code of rank 1 if there is a unit in at least one coordinate. Hence there are $4^r - 2^r$ vectors that generate a free code of rank 1 in C, namely we subtract the number of vectors with no units from the total number. Any of these vectors has 4 vectors in the code it generates. To choose the second vector we eliminate all vectors that are a multiple of the first that have at least one coordinate with a unit. Out of the 4 vectors that are multiples $\phi(4) = 4-2$ have such a coordinate because multiplication by a non-unit makes each coordinate a non-unit. Hence there are

$$4^r - 2^r - 2 = 4^r - 2^r - (4 - 2)$$

ways to pick two vectors that generate a free code.

Continuing in this manner we see that to pick h vectors to generate a free code there are

(15)
$$\prod_{j=0}^{h-1} (4^r - 2^r - (4^j - 2^j)) =$$

$$(4^r - 2^r)(4^r - 2^r - (4 - 2)) \dots (4^r - 2^r - (4^{h-1} - 2^{h-1}))$$

ways of generating a free code of rank h.

Next we must divide by the number of different ways each subcode of rank h can be generated, which is

(16)
$$\prod_{j=0}^{h-1} (4^h - 2^h - (4^j - 2^j)) =$$

$$(4^r - 2^r)(4^r - 2^r - (4 - 2)) \dots (4^r - 2^r - (4^{h-1} - 2^{h-1})).$$

This gives the following.

Lemma 2.1 *Let C be a free code of rank r over $\mathbb{Z}_4$ then the number of free subcodes of rank h is*

$$\left[\begin{array}{c} r \\ h \end{array}\right]_4 = \frac{\prod_{j=0}^{h-1}(4^r - 2^r - (4^j - 2^j))}{\prod_{j=0}^{h-1}(4^h - 2^h - (4^j - 2^j))} \tag{17}$$

Now assume C is a free code with rank r. If D is a subcode of type 2^h, then $D = 2D'$ where D' is a free code of rank h, note D and D' generally do not have the same number of elements. This implies the following.

Let C be a code of type $\{k_1, k_2\}$. We shall count the number of subcodes of type $\{h_1, h_2\}$. The number of vectors in C with at least one unit in C is $(4^{k_1} - 2^{k_1})2^{k_2}$ since any of the vectors with a unit generated by the free part plus a vector in the non-free part will have a unit in at least one coordinate. Choose a vector from this and call it v_1.

There are two multiples of v_1 that have at least one unit so there are $(4^{k_1}-2^{k_1})2^{k_2}-(4-2)$ choices for v_2. There are 4^2 vectors in $\langle v_1, v_2 \rangle$ and $4^2 - 2^2$ have a unit in them. Hence the number of ways to chose h_1 vectors to generate a free code is:

$$A(h_1) = \prod_{j=0}^{h_1-1} (4^{k_1} - 2^{k_1})2^{k_2} - (4^j - 2^j) \tag{18}$$

The number of ways each of these spaces can be generated is

$$B(h_1) = \prod_{j=0}^{h_1-1} (4^{h_1} - 2^{h_1} - (4^j - 2^j)). \tag{19}$$

Note that this is not simply the same as above with a different index as it is for codes over fields.

Next we must choose h_2 vectors with no units that are not in $\langle v_1, \ldots, v_{h_1} \rangle$. There are $4^{k_1} - (4^{k_1} - 2^{k_1}) = 4^{k_1} - 4^{k_1} + 2^{k_1} = 2^{k_1}$ vectors with no units in the free part and 2^{k_2} vectors with no units in the non-free part. Hence there are $2^{k_1+k_2}$ vectors with no units, since the sum of any two vectors with no units is a vector with no units. Then we proceed as usual and we have $\left[\begin{array}{c} k_1 + k_2 \\ h_2 \end{array}\right]_2$ different ways of getting h_2 vectors for the non-free part divided by the number of ways each space is generated.

For a type $\{k_1, k_2\}$ code C over $\mathbb{Z}_4$ let $\nu(C, h_1, h_2)$ denote the set of subcodes $[n; h_1, h_2]$, $h_1 \leq k_1, h_1 + h_2 \leq k_1 + k_2$ of C. Note that while it is necessary for $h_1 \leq k_1$ it is not necessary that h_2 be less than k_2. For example a code of type $\{0, 1\}$ has a subcode of type $\{0, 1\}$. Of course, it is true that $h_1 + h_2 \leq k_1 + k_2$. We denote by $\mu(k_1, k_2, h_1, h_2) = |\nu(C, h_1, h_2)|$.

Theorem 2.2 *Let C be a linear code over $\mathbb{Z}_4$ of type $\{k_1, k_2\}$ then the number of subcodes of type $\{h_1, h_2\}$ is given by*

$$\frac{A(h_1)}{B(h_1)} \left[\begin{array}{c} k_1 + k_2 \\ h_2 \end{array}\right]_2 \tag{20}$$

Now we consider the Lee support weights for a linear code over $\mathbb{Z}_4$. We have the following theorem.

Theorem 2.3 *Let C be a linear code of length n over $\mathbb{Z}_4$ with type $4^{k_1}2^{k_2}$. Then we have*

$$\begin{aligned} \mathrm{wt}_L(C) &= \frac{1}{4^{k_1-1}2^{k_2}} \sum_{\boldsymbol{x}\in C} (\text{L-wt}(\boldsymbol{x}) - \mathrm{wt}(\boldsymbol{x})) \\ &= \frac{1}{4^{k_1-1}2^{k_2}} \sum_{\boldsymbol{x}\in C} |\{i : \ x_i = 2\}|. \end{aligned}$$

Proof. In the array $A(C)$, let n_0 be the number of columns in which 0 and 2 are balanced and let n_1 be the number of columns in which 0,1,2 and 3 occurs equally often. So we have $2n_0 + n_1 = \mathrm{wt}_L(C)$. Hence we have

$$\begin{aligned} \sum_{\boldsymbol{x}\in C} (\text{L-wt}(\boldsymbol{x}) - \mathrm{wt}(\boldsymbol{x})) &= (n_0(|C|/2\cdot 2) + n_1(|C|/4\cdot 1 + |C|/4\cdot 2 + |C|/4\cdot 1)) \\ &\quad -(n_0(|C|/2\cdot 1) + n_1(|C|/4\cdot 1 + |C|/4\cdot 1 + |C|/4\cdot 1)) \\ &= |C|/4((4n_0 + 4n_1) - (2n_0 + 3n_1)) \\ &= |C|/4\cdot \mathrm{wt}_L(C) \end{aligned}$$

□

Using the above theorem we obtain a connection between the Lee support weight of C and the Lee support weights of its subcodes.

Theorem 2.4

$$\begin{aligned} \sum_{D\in\nu(C,h_1,h_2)} \mathrm{wt}_L(D) &= \frac{1}{2^{2h_1-2+h_2}}(2^{2k_1-2+k_2}\mathcal{A}_1\mathrm{wt}_L(C) + 2^{k_1+k_2-2}(\mathcal{A}_3 - \mathcal{A}_1)\mathrm{wt}_L(\tilde{C}) \\ &\quad +2^{k_1-2}(\mathcal{A}_2 - \mathcal{A}_3)\mathrm{wt}_L(\hat{C})), \end{aligned}$$

where $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ are numbers given in Proposition 6 *of* [1].

Proof. The proof is on the line of the proof of Proposition 6 of [1]. □

3 MacWilliams Relations

We define the following weight enumerator which is a natural generalization of the joint weight enumerator for codes over $\mathbb{Z}_k$. Let $C_1, C_2, .., C_g$ be codes such that C_i is a code in R. The complete joint weight enumerator of genus g for codes $C_1, \ldots, C_g$ of length n is defined as

(21) $$\mathfrak{J}_{C_1,C_2,\ldots,C_g}(X_{\mathbf{a}} \text{ with } \mathbf{a}\in R^g) = \sum_{(c_1,c_2,\ldots,c_g)\in C_1\times C_2\times\ldots\times C_g} \prod_{\mathbf{a}\in\mathbb{Z}_k^g} X_{\mathbf{a}}^{n_{\mathbf{a}}(c_1,c_2,\ldots,c_g)}$$

where $n_{\mathbf{a}}(c_1, c_2, \ldots, c_g) = |\{j|((c_1)_j, (c_2)_j, \ldots, (c_g)_j) = \mathbf{a}\}|$, and $c_i = ((c_i)_1, \ldots, (c_i)_n)$.

We shall describe the matrix we need to produce the MacWilliams relations for codes over an arbitrary commutative ring.

We want the orthogonality given by the character group associated to the additive group of the ring to match the given inner product, where the orthogonality given by the character group is:

$$\chi(C) = \{v \mid \chi_v(w) = 1 \forall w \in C\}$$

where $\chi_v \in \widehat{G}$.

Let σ be a character in $\widehat{G}$ associated with the element 1. For $a, b \in R$ set $\chi_b(a) = \sigma(a\overline{b})$. We see that χ_b is a character associated with the element b of the ring.

This gives

$$\prod \chi_{w_i}(\overline{v_i}) = \prod \sigma(v_i \overline{w_i}) = \sigma(\sum v_i \overline{w_i})$$

If $(\sum v_i \overline{w_i}) = 0$ then $\sigma(\sum v_i \overline{w_i}) = 1$.

It is shown in [6] that the matrix produced by these characters gives the MacWilliams relation for the complete weight enumerator, where the complete weight enumerator for a code C is

$$W_C(x_0, \ldots, x_{r-1}) = \sum_{c \in C} A_{a_0, \ldots, a_{r-1}} x_0^{a_0} x_1^{a_1} \ldots x_{r-1}^{a_{r-1}}$$

where there are a_i coordinates in c with i in the coordinate.

The matrix is defined by

$$T_{\alpha_i, \alpha_j} = \chi_{\alpha_j}(\overline{\alpha_i}). \tag{22}$$

Theorem 3.1 *Let C be a linear code over a finite ring R, with $|R| = k$, then*

$$W_{C^{\perp}}(x_0, \ldots, x_{k-1}) = \frac{1}{|C|} W_C(T(x_0, \ldots, x_{k-1})).$$

For a complete description, see [6].

For $\mathbb{Z}_k$ the matrix T is given as follows. Let η be a complex k-th root of unity, and define the index the matrix T with the elements of $\mathbb{Z}_k$ defining $T_{i,j} = \eta^{ij}$.

The MacWilliams relations for the joint weight enumerator over $\mathbb{Z}_k$ were corrected in [5]. They can be generalized to the following lemma.

Lemma 3.2 *Let $C_1, C_2, \ldots, C_g$ be linear codes in R and let $\tilde{C}$ denote either C or $C^{\perp}$. Then*

$$\mathfrak{J}_{\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_g}(X_{\mathbf{a}}) = \frac{1}{\prod_{i=1}^{g} |C_i|^{\delta_{\tilde{C}_i}}} \cdot (\otimes_{i=1}^{g} T^{\delta_{\tilde{C}}}) \mathfrak{J}_{C_1, \ldots, C_g}(X_{\mathbf{a}}) \tag{23}$$

where

$$\delta_{\tilde{C}} = \begin{cases} 0 & \text{if } \tilde{C} = C, \\ 1 & \text{if } \tilde{C} = C^{\perp}. \end{cases}$$

Note that the matrix $\otimes_{i=1}^{g} T^{\delta_{\tilde{C}}}$ is an $|R|^g$ by $|R|^g$ matrix and that $\mathfrak{J}_{\tilde{C_1},\tilde{C_2},\ldots,\tilde{C_g}}(X_{\mathbf{a}})$ is a polynomial in $|R|^g$ variables. The proof of this lemma is given in the preprint [7].
Denote by $\mathfrak{J}(C,g)(X_{\mathbf{a}}) = \mathfrak{J}_{C_1,C_2,\ldots,C_g}(X_{\mathbf{a}})$ with $C_i = C$ for $i = 1,\ldots,g$.

Let $\mathfrak{A}_{g,\mathbf{h}} = \{\mathbf{j}$ such that a subcode of type $\mathbf{j}$ can be generated from a type $\mathbf{h}$ code using g (not necessarily independent) vectors$\}$, where $\mathbf{h} = \{h_1, h_2, \ldots, h_s\}$ and $\mathbf{j} = \{j_1, j_2, \ldots, j_s\}$

Lemma 3.3 *Let C be a linear code over R then*

$$\mathfrak{J}(C,g)(X_{\mathbf{a}}) = \sum_{\mathbf{j}\in\mathfrak{A}_{g,\mathbf{h}}} \Psi(g,\mathbf{h},\mathbf{j}) W_C^{\mathbf{j}}(X_{\mathbf{0}} = x, X_{\mathbf{a}} = y(\mathbf{a} \neq 0)) \tag{24}$$

where $\Psi(g,\mathbf{h},\mathbf{j})$ denotes the number of ways a subcode of type $\mathbf{j}$ can be generated from a subspace of type $\mathbf{h}$ using g vectors.

Proof. Given a set of g vectors represented by $X_{\mathbf{a}}$, then the number of $X_{\mathbf{a}}$ that are not 0 is equal to the support of the space generated by the vectors. Moreover, each subspace is generated $\Psi(g,\mathbf{h},\mathbf{j})$ different times. □

Note that a similar thing cannot be done by simply considering ranks because from knowing only the rank of a code it is not possible to determine how many subcodes of a given rank there are. For example, a rank 1 code over $\mathbb{Z}_6$ may have a subcode of rank 1 or it may not, depending on whether the code is $\mathbb{Z}_6$ or $\{0,3\}$.

This lemma allows us to generate MacWilliams relations for the higher weight enumerators.

Theorem 3.4 *Let C be a linear code over R, $|R| = k$, then*

$$\sum_{\mathbf{j}\in\mathfrak{A}_{g,\mathbf{h}}} \Psi(g,\mathbf{h},\mathbf{j}) W_{C^\perp}^{\mathbf{j}}(x,y) = \frac{1}{|C|^g} \sum_{\mathbf{j}\in\mathfrak{A}_{g,\mathbf{h}}} \Psi(g,\mathbf{h},\mathbf{j}) W_{C^\perp}^{\mathbf{j}}(x + (k^g - 1)y, x - y) \tag{25}$$

Proof. Specializing the variables collapses the matrix $\otimes_{i=1}^{g} T$, the first row of which is all 1 and hence collapses to $k^g - 1$.

Every other row has a 1 in the first column and then noticing that $\sum_{a\in R} \chi_b(a) = 0$, so summing all but the first row gives -1. Hence the matrix becomes

$$\begin{pmatrix} 1 & k^g - 1 \\ 1 & -1 \end{pmatrix} \tag{26}$$

□

A similar technique was used for codes over fields in [8].

Example: Let C be the linear code of length 2 over $\mathbb{Z}_4$ generated by $(1,0)$ and $(0,2)$. The code has type $\{1,1\}$. We have

$$\begin{aligned} J(C,2)(x,y,\ldots,y) &= W^{0,0}(x,y)+12W^{1,0}(x,y) \\ &+ 3W^{0,1}(x,y)+6W^{0,2}(x,y)+24W^{1,1}(x,y), \end{aligned}$$

where $W^{0,0}(x,y) = x^2$, $W^{1,0}(x,y) = xy+y^2$, $W^{0,1}(x,y) = 2xy+y^2$, $W^{0,2}(x,y) = y^2$, $W^{1,1}(x,y) = y^2$. Then

$$\begin{aligned} \frac{1}{64}(W^{0,0}(x+15y,x-y) &+ 12W^{1,0}(x+15y,x-y)+3W^{0,1}(x+15y,x-y) \\ &+ 6W^{0,2}(x+15y,x-y)+24W^{1,1}(x+15y,x-y) \\ &= x^2+3xy. \end{aligned}$$

Now, $C^\perp = \{(0,0),(0,2)\}$ and is of type $\{0,1\}$, with $W^{0,0}(x,y) = x^2$, $W^{0,1}(x,y) = xy$ and $J(C,2)(x,y,\ldots,y) = W^{0,0}(x,y)+3W^{0,1}(x,y)$.

Notice also that

$$\begin{aligned} W_C(x,y) = J(C,1)(x,y) &= x^2+4xy+3y^2 \\ &= W^{0,0}(x,y)+2W^{1,0}(x,y)+W^{0,1}(x,y) \\ &= x^2+2(xy+y^2)+(2xy+y^2) \end{aligned}$$

as expected by Theorem 1.1.

4 Bounds

4.1 A Singleton Bound

A *chain ring* R is a finite ring with Jacobson radical $J(R) \neq 0$ whose principal left ideals form a chain (see [22]). It follows easily that $\mathbb{Z}_{p^m}$ is a kind of chain ring, where p is a prime. In [18], Horimoto and Shiromoto proved the following Singleton type bound for GHWR of linear codes over finite chain rings:

Proposition 4.1 *Let C be a linear code of length n over a finite chain ring R. For any r, $1 \leq r \leq \text{rank}(C)$, we have*

$$d_r^H(C) \leq n - \text{rank}(C) + r.$$

In this subsection we shall find the corresponding Singleton bound for the higher weights over a kind of non-chain rings $\mathbb{Z}_k$.

The Chinese Remainder Theorem was used in [10] to form MDR codes over $\mathbb{Z}_k$, here we recall the basic definitions and a few facts. Let k and q be integers with q dividing k, define the map

$$\Psi_q : (\mathbb{Z}/k\mathbb{Z})^n \to (\mathbb{Z}/q\mathbb{Z})^n$$

by

$$\Psi_q(\alpha_1, \alpha_2, \cdots, \alpha_n) = (\alpha_1 \pmod q), \alpha_2 \pmod q), \cdots, \alpha_n \pmod q))$$

where $v = (\alpha_1, \alpha_2, \cdots, \alpha_n)$.

If k is a positive integer with $k = \prod_{i=1}^{s} q_i$ and $\gcd(q_i, q_j) = 1$ then define the map

$$\Psi : (\mathbb{Z}/k\mathbb{Z})^n \to (\mathbb{Z}/q_1\mathbb{Z})^n \times (\mathbb{Z}/q_2\mathbb{Z})^n \times \cdots \times (\mathbb{Z}/q_s\mathbb{Z})^n$$

by

$$\Psi(\boldsymbol{v}) := (\Psi_{q_1}(\boldsymbol{v}), \Psi_{q_2}(\boldsymbol{v}), \cdots, \Psi_{q_s}(\boldsymbol{v})).$$

If $C^{(q_1)}, C^{(q_2)} \cdots C^{(q_s)}$ be codes of length n, with $C^{(q_i)}$ a code over $\mathbb{Z}_{q_i}$, define the Chinese product by

$$\mathrm{CRT}(C^{(q_1)}, C^{(q_2)}, \cdots, C^{(q_s)}) = \{\Psi^{-1}(\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_s) \mid \boldsymbol{v}_i \in C^{(q_i)}\},$$

where $\Psi^{-1}(\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_s)$ is the unique vector in $(\mathbb{Z}/\mathbb{Z}_k)^n$ that is congruent component wise to $\boldsymbol{v}_i \pmod{q_i}$.

The generalized Chinese Remainder Theorem implies that CRT is the inverse image of the map Ψ.

We have the following fact. Let $C^{(q_1)}, C^{(q_2)}, \cdots, C^{(q_s)}$ be codes over $\mathbb{Z}_{q_1}, \mathbb{Z}_{q_2}, \cdots, \mathbb{Z}_{q_s}$ respectively. Then

$$\mathrm{rank}(\mathrm{CRT}(C^{(q_1)}, C^{(q_2)}, \cdots, C^{(q_s)})) = \mathrm{Max}\{\mathrm{rank}(C^{(q_i)})\}.$$

Additionally we can see that if $C = (\mathrm{CRT}(C^{(q_1)}, C^{(q_2)}, \cdots, C^{(q_s)}))$ and D is a subcode of rank h of C then

$$D = \mathrm{CRT}(D^{(q_1)}, D^{(q_2)}, \cdots, D^{(q_s)}))$$

where $D^{(q_i)} \subseteq C^{(q_i)}$ and $\mathrm{Max}\{\mathrm{rank}(D^{(q_i)})\}$ is h.

Lemma 4.2 *Let* $C = (\mathrm{CRT}(C^{(q_1)}, C^{(q_2)}, \cdots, C^{(q_s)}))$ *then* $d_g(C) = \mathrm{Min}\{d_g^H(C^{(q_i)})\}$.

Proof. This follows from the fact that $D = \mathrm{CRT}(\mathbf{0}, \ldots, D^{(q_i)}, \ldots, \mathbf{0}, \ldots))$ is an R-submodule of C of rank g for all i if $D^{(q_i)}$ has rank g. □

Theorem 4.3 *Let C be a linear code of length n over $\mathbb{Z}_k$ of rank r. Then*

$$d_g(C) \leq n - r + g,$$

for any h, $1 \leq g \leq r$.

Proof. Follows directly from Proposition 4.1 and Lemma 4.2. □

We shall call codes meeting this bound as *g-th Maximum Hamming Distance Separable with respect to Rank* (g-th MHDR) codes.

The following theorem and proof is similar to that for MDR codes given in [10].

Theorem 4.4 *Let $C^{(k_1)}, C^{(k_2)}, \cdots, C^{(k_s)}$ be codes over $\mathbb{Z}_{k_1}, \mathbb{Z}_{k_2}, \ldots, \mathbb{Z}_{k_s}$ respectively. If $C^{(k_i)}$ is an g-th MHDR code for all i (not necessary the same rank), then $\mathrm{CRT}(C^{(k_1)}, C^{(k_2)}, \cdots, C^{(k_s)})$ is a g-th MHDR code.*

Proof. Let $C := \mathrm{CRT}(C^{(k_1)}, C^{(k_2)}, \cdots, C^{(k_s)})$. We have $\mathrm{rank}(C) = \mathrm{Max}\{\mathrm{rank}(C^{(k_i)})\}$. So

$$\begin{aligned} d_g^H(C) &= \min\{d_g^H(C^{(k_i)})\} = \min\{n - \mathrm{rank}(C^{(k_i)}) + g\} \\ &= n - \mathrm{Max}\{\mathrm{rank}(C^{(k_i)})\} + g = n - \mathrm{rank}(C) + g. \end{aligned}$$

□

4.2 Bounds for GLWR

In this section, we give some bounds for GLWR of linear codes over $\mathbb{Z}_4$.

Lemma 4.5 *If C is a linear code of length n over $\mathbb{Z}_4$ with $\mathrm{rank}(C) = 2$, then there exists a codeword $\mathbf{0} \neq \boldsymbol{v} \in C$ such that $\text{L-wt}(\boldsymbol{v}) \leq \mathrm{wt}_L(C)$.*

Proof. We assume that C is generated by $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$, where both x_i and y_i are not 0 for any i. If either x_i or y_i is 1 or 3 and the other is 0 or 2, then the Lee weight of $\alpha x_i + \beta y_i$ are at most 1 for any units α, β in $\mathbb{Z}_4$. And if $2x_i = 2y_i = 0$, then the Lee weight of $\alpha x_i + \beta y_i$ are at most 2 for any units α, β in $\mathbb{Z}_4$. So if $|\{i \mid x_i = y_i = 1 \text{ or } 3\}| \leq |\{i \mid \{x_i, y_i\} = \{1, 3\} \text{ or } \{3, 1\}\}|$ (Resp., $|\{i \mid x_i = y_i = 1 \text{ or } 3\}| \geq |\{i \mid \{x_i, y_i\} = \{1, 3\} \text{ or } \{3, 1\}\}|$), then $\text{L-wt}(\boldsymbol{x} + \boldsymbol{y}) \leq \mathrm{wt}_L(C)$ (Resp., $\text{L-wt}(\boldsymbol{x} + 3\boldsymbol{y}) \leq \mathrm{wt}_L(C)$). The lemma follows. □

Theorem 4.6 *Let C be a linear code of length n over $\mathbb{Z}_4$ with $\mathrm{rank}(C) \geq 2$. Then we have $1 \leq d_1^L(C) \leq d_2^L(C)$.*

Proof. Let D be a submodule of C with $\mathrm{wt}_L(D) = d_2^L(C)$ and $\mathrm{rank}(D) = 2$. From Lemma 4.5, there exists a codeword $\mathbf{0} \neq \boldsymbol{v} \in D$ such that $\text{L-wt}(\boldsymbol{v}) \leq \mathrm{wt}_L(D)$. Since $d_1^L(C) \leq \text{L-wt}(\boldsymbol{v})$, the theorem follows. □

The following monotonicity is well-known for a linear code C of rank k over a chain ring ([18] and [27]):

$$1 \leq d_1^H(C) < d_2^H(C) < \cdots < d_k^H(C) \leq n.$$

Based on the above inequality, with respect to the GLWR, we had conjectured as follows for a linear code C of length n over $\mathbb{Z}_4$ with $\mathrm{rank}(C) = k > 0$:

$$1 \leq d_1^L(C) \leq d_2^L(C) \leq \cdots \leq d_k^L(C) \leq 2n.$$

However, Hashimoto ([14]) recently proved a counter-example of the conjecture.

Example 4.7 ([14]) Let C be a linear code of length 21 over $\mathbb{Z}_4$ having a generator matrix

$$G = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 2 & 1 & 2 & 0 & 1 & 0 & 1 & 1 & 0 & 3 & 2 & 1 & 1 & 3 & 1 & 1 \\ 0 & 1 & 0 & 2 & 1 & 0 & 0 & 1 & 2 & 1 & 1 & 0 & 3 & 1 & 0 & 1 & 2 & 1 & 1 & 3 & 1 \\ 0 & 0 & 1 & 0 & 2 & 1 & 2 & 0 & 1 & 0 & 1 & 1 & 0 & 3 & 1 & 1 & 1 & 2 & 1 & 1 & 3 \end{pmatrix}.$$

Then it follows that $d_2^L(C) = 22$ and $d_3^L(C) = 21$. Therefore it shows that the conjecture is false and this is a counter-example of a code whose lengths are a minimum.

Now, we give a Singleton type bound on the GLWR.

Theorem 4.8 *For a linear code C of length n over $\mathbb{Z}_4$ and any r, $1 \leq r \leq \mathrm{rank}(C)$,*

$$\left\lfloor \frac{d_r^L(C) - 2r + 1}{2} \right\rfloor \leq n - \mathrm{rank}(C).$$

Proof. We set $d_r^L := d_r^L(C)$ and $k := \mathrm{rank}(C)$. Now, we assume that

$$\left\lfloor \frac{d_r^L - 2r + 1}{2} \right\rfloor > n - k. \tag{27}$$

Note that

$$\left\lfloor \frac{d_r^L - 2r + 1}{2} \right\rfloor = \begin{cases} (d_r^L - 2r)/2 & d_r^L : \text{even} \\ (d_r^L - 2r + 1)/2 & d_r^L : \text{odd.} \end{cases}$$

If d_r^L is even, then the bound (27) is $d_r^L > 2n - 2k + 2r$. On the other hand, from (14) and Proposition 4.1, we have

$$d_r^L \leq 2n - 2k + 2r. \tag{28}$$

A contradiction.

If d_r^L is odd, then the bound (27) is $d_r^L > 2n-2k+2r-1$. Thus we have $d_r^L = 2n-2k+2r$ from (28). This contradicts that d_r^L is odd. Therefore the theorem follows. □

Remark 4.9 In [9] and [24], it is shown that for a linear code C of length n over $\mathbb{Z}_4$ with minimum Lee weight d_L,

$$\left\lfloor \frac{d_L - 1}{2} \right\rfloor \leq n - \text{rank}(C).$$

Since $d_L = d_1^L(C)$, the bound in Theorem 4.8 is a generalization of the above bound.

If a linear code C of length n over $\mathbb{Z}_4$ meets the bound in Theorem 4.8 for r, that is, $\left\lfloor (d_r^L(C) - 2r + 1)/2 \right\rfloor = n - \text{rank}(C)$, then we shall call the code C as *r-th Maximum Lee Distance Separable with respect to Rank* (r-th MLDR) code. Now we shall give a connection between r-th MLDR codes and r-th MHDR codes.

Lemma 4.10 *If C is an r-th MLDR code, then $d_r^L(C) = 2d_r^H(C) - 1$ or $2d_r^H(C)$.*

Proof. Since C is an r-th MLDR code, we have

$$\left\lfloor \frac{d_r^L(C) - 2r + 1}{2} \right\rfloor = n - \text{rank}(C). \tag{29}$$

We assume that $d_r^L(C) < 2d_r^H(C) - 1$. If $d_r^L(C)$ is odd, then we have the following equation from (29):

$$d_r^L(C) = 2n - 2\text{rank}(C) + 2r - 1.$$

Since $d_r^L(C) < 2d_r^H(C) - 1$, we have

$$2n - 2\text{rank}(C) + 2r - 1 < 2d_r^H(C) - 1 \Longleftrightarrow n - \text{rank}(C) + r < d_r^H(C).$$

A contradiction from the bound in Proposition 4.1. In the case that $d_r^L(C)$ is even, the proof follows. □

Theorem 4.11 *Let C be a linear code C of length n over $\mathbb{Z}_4$. If C is an r-th MLDR code, then C is an r-th MHDR code.*

Proof. From the above lemma, we have $d_r^L(C) = 2d_r^H(C) - 1$ or $2d_r^H(C)$. In both case,

$$n - \text{rank}(C) = \left\lfloor \frac{d_r^L(C) - 2r + 1}{2} \right\rfloor = d_r^H(C) - r.$$

□

Theorem 4.12 *Let C be an r-th MHDR code of length n over $\mathbb{Z}_4$. C is an r-th MLDR code if and only if $d_r^L(C) = 2d_r^H(C) - 1$ or $2d_r^H(C)$.*

Proof. Since C is an r-th MLDR code if and only if

$$\left\lfloor \frac{d_r^L(C) - 2r + 1}{2} \right\rfloor = d_r^H(C) - r.$$

If $d_r^L(C)$ is odd, then

$$\left\lfloor \frac{d_r^L(C) - 2r + 1}{2} \right\rfloor = \frac{d_r^L(C) - 2r + 1}{2} = d_r^H(C) - r,$$

and if $d_r^L(C)$ is even, then

$$\left\lfloor \frac{d_r^L(C) - 2r + 1}{2} \right\rfloor = \frac{d_r^L(C) - 2r}{2} = d_r^H(C) - r.$$

The theorem follows. □

It is known that if C is a linear code of length n over $\mathbb{Z}_4$ with minimum Hamming weight d_H and minimum Lee weight d_L, then

$$d_H \geq \left\lceil \frac{d_L}{2} \right\rceil \tag{30}$$

(cf. [23]). In [25], they have proved the following Griesmer type bound for linear codes over finite quasi-Frobenius rings.

Lemma 4.13 *Let C be a linear code of length n over $\mathbb{Z}_4$ with* $\mathrm{rank}(C) = k$ *and minimum Hamming weight d_H. Then*

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d_H}{2^i} \right\rceil.$$

Using (30) and Lemma 4.13, we have the following Griesmer type bound for minimum Lee weights of linear codes over $\mathbb{Z}_4$.

Proposition 4.14 *Let C be a linear code of length n over $\mathbb{Z}_4$ with* $\mathrm{rank}(C) = k$ *and minimum Lee weight d_L. Then*

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{\lceil d_L/2 \rceil}{2^i} \right\rceil.$$

Now we have a generalized Griesmer type bound for GLWR.

Theorem 4.15 *For a linear code C of length n over $\mathbb{Z}_4$ and any r, $1 \leq r \leq \mathrm{rank}(C)$, we have*

$$d_r^L(C) \geq \sum_{i=0}^{r-1} \left\lceil \frac{\lceil d_1^L(C)/2 \rceil}{2^i} \right\rceil.$$

Proof. For a $\mathbb{Z}_4$-submodule D of C with $\mathrm{wt}_L(D) = d_r^L(C)$ and $\mathrm{rank}(D) = r$, let D' be the code having a generator matrix obtained from a generator matrix of D by deleting the zero columns. Since the length of D' is less than or equal to $\mathrm{wt}_L(D)$ and the minimum Lee weight of D' is greater than or equal to $d_1^L(C)$, the theorem follows from Proposition 4.14. □

Let C be a linear code C of length n over $\mathbb{Z}_4$. From the definitions of GLWR and GHWR, we have

$$d_r^H \geq \left\lceil \frac{d_r^L}{2} \right\rceil \tag{31}$$

for any r. It is known that if C is a linear code C of length n over $\mathbb{Z}_4$ with $\mathrm{rank}(C) = k$ and minimum Hamming weight d_H, then $\mathrm{Soc}(C)$ is isomorphic to a binary $[n, k, d]$ code (cf. [18]).

Lemma 4.16 ([18]) *For any r, $1 \leq r \leq \mathrm{rank}(C)$, we have*

$$d_r^H(C) = d_r^H(\mathrm{Soc}(C)).$$

Using the above lemma and Theorem 3.19 (p. 35 in [13]), the lemma follows:

Lemma 4.17 *Let C be a linear code C of length n over $\mathbb{Z}_4$ with $\mathrm{rank}(C) = k$. Then*

$$n \geq d_r^H(C) + \sum_{i=1}^{k-r} \left\lceil \frac{d_r^H(C)}{2^i(2^i-1)} \right\rceil,$$

for any r, $1 \leq r \leq k$.

Now we have a generalized Griesmer type bound for GLWR.

Theorem 4.18 *Let C be a linear code C of length n over $\mathbb{Z}_4$ with $\mathrm{rank}(C) = k$. Then*

$$n \geq \left\lceil \frac{d_r^L(C)}{2} \right\rceil + \sum_{i=1}^{k-r} \left\lceil \frac{\left\lceil d_r^L(C)/2 \right\rceil}{2^i(2^i-1)} \right\rceil,$$

for any r, $1 \leq r \leq k$.

Proof. The theorem follows from the above lemma and the inequality (31). □

Let C be a free linear code of length n over $\mathbb{Z}_4$ with $\mathrm{rank}(C) = r$ and minimum Lee weight d_L then the following Griesmer type bound is known [1].

Lemma 4.19

$$n \geq \sum_{i=0}^{r-1} \left\lceil \frac{3 \cdot 2^{i(i-1)}/2}{4 \cdot \Pi_{j=0}^{i-1}(2^{i+1-j}+1)} d_L \right\rceil.$$

Thus we have the following bound for the free codes. This is better than the bound given by the Theorem 4.15 for free codes. Its proof is similar.

Theorem 4.20

$$d_r^L(C) \geq \sum_{i=0}^{r-1} \left\lceil \frac{3 \cdot 2^{i(i-1)}/2}{4 \cdot \Pi_{j=0}^{i-1}(2^{i+1-j}+1)} d_L \right\rceil.$$

It is known that the octacode meets the bound of the Lemma 4.19. It will be interesting to construct codes over $\mathbb{Z}_4$ that meets the above bound of Theorem 4.20. However except for $r = 1$ the octacode meet the above bound for GLWR (see Theorem 5.7).

5 Determination of Generalized Weight

In this section we look at the Generalized weights for some well known classes of codes. Let C be a linear code over $\mathbb{Z}_4$ of length n and 2-dimension k. For $\boldsymbol{x} \in C$ let $\omega_2(\boldsymbol{x}) = |\{i \mid x_i = 2\}|$. The following remark follows from Theorem 2.3.

Remark 5.1 *For* $1 \leq r \leq k$,

$$d_r^L(C) = \frac{1}{2^{r-2}} \min \left\{ \sum_{\boldsymbol{x} \in D} \omega_2(\boldsymbol{x}) \mid D : [n, r] \text{ subcode of } C \right\}.$$

It is clear from the above remark 5.1 that it is difficult to find the generalized Lee weight since $\omega_2(\boldsymbol{x})$ is not a metric. Now we find the generalized Lee weight for the several known classes of the codes.

The following lemma follows from definition.

Lemma 5.2 *Let C be a linear code over $\mathbb{Z}_4$ with generator matrix $G = [2g_1, 2g_2, \ldots, 2g_k]$ then for $1 \leq r \leq k$ we have*

$$d_r^L(C) = 2d_r^H(C)$$

where $d_r^H(C)$ is the Hamming weight hierarchy of C.

5.1 First-Order Reed Muller Code

The first order Reed Muller code $R^{1,m}$ over $\mathbb{Z}_4$ is a code of length $n = 2^{m-1}$, rank m, 2-dimension $m+1$ with minimum Hamming weight 2^{m-2} and minimum Lee weight 2^{m-1}.

Theorem 5.3 *The Lee weight hierarchy of $R^{1,m}$ with respect to 2-dimension is given by* 2-$d_r^L = 2^{m-r}(2^r - 1), 1 \leq r \leq m-1$, 2-$d_m^L = 2^m$ *and* 2-$d_{m+1}^L = 2^{m-1}$.

Proof. It follows by Lemma 5.2 (see [11]).

□

Remark 5.4 Note that the monotonicity fails for GLWT as in Theorem 5.3 2-$d_m^L >$ 2-d_{m+1}^L.

5.2 Simplex Codes

The Hamming weight hierarchy of quaternary simplex codes of type α and β with respect to 2-dimension were studied in [4]. The next theorem finds the Hamming weight hierarchy with respect to rank. Note that the rank of both the simplex codes is k.

Theorem 5.5 *The Hamming weight hierarchy of S_k^α and S_k^β with respect to rank is given by*

$$d_r^H\left(S_k^\alpha\right) = 2d_r^H\left(S_k^\beta\right) = 2^{2k-r}(2^r - 1), 1 \leq r \leq k.$$

Proof. We will prove it only for S_k^β, since the other case is similar. By Lemma 4.16 and Lemma 5 of [4] the result follows.

□

5.3 Quaternary Golay Code

The quaternary lifted Golay code has length 24, rank 12, 2-dimension 24, minimum Hamming weight 8 and minimum Lee weight 12.

Theorem 5.6 *The quaternary Golay code QR_{24} has Lee weight hierarchy (with respect to rank)* $\{12, 14, 16, 16, 17, 18, 19, 20, 21, 22, 23, 24\}$.

Proof. It is a starigthforward computation.

□

5.4 Octacode

The octacode QR_8 is a code over $\mathbb{Z}_4$ of length 8, 2-dimension 8, minimum Hamming weight 4 and minimum Lee weight 6.

Theorem 5.7 *The quaternary octacode QR_8 has Lee weight hierarchy (with respect to rank)* $\{6, 6, 7, 8\}$.

Proof. Straightforward.

□

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