On Senary Simplex Codes

Manish K. Gupta[1], David G. Glynn[1], and T. Aaron Gulliver[2]

[1] Department of Mathematics, University of Canterbury,
Private Bag 4800, Christchurch, New Zealand
m.k.gupta@ieee.org
d.glynn@math.canterbury.ac.nz

[2] Department of Electrical and Computer Engineering, University of Victoria,
P.O. Box 3055, STN CSC, Victoria, B.C., Canada V8W 3P6
agullive@engr.uvic.ca

Abstract. This paper studies senary simplex codes of type α and two punctured versions of these codes (type β and γ). Self-orthogonality, torsion codes, weight distribution and weight hierarchy properties are studied. We give a new construction of senary codes via their binary and ternary counterparts, and show that type α and β simplex codes can be constructed by this method.

1 Introduction

There has been much interest in codes over finite rings in recent years, especially the rings $\mathbb{Z}_{2k}$ where $\mathbb{Z}_{2k}$ denotes the ring of integers modulo $2k$. In particular codes over $\mathbb{Z}_4$ have been widely studied [1], [5],[6],[7],[8], [9],[10],[11], [12]. More recently $\mathbb{Z}_4$-simplex codes (and their Gray images), have been investigated by Bhandari, Gupta and Lal in [2]. Good binary linear and non-linear codes can be obtained from codes over $\mathbb{Z}_4$ via the Gray map. Thus it is natural to investigate simplex codes over the ring $\mathbb{Z}_{2k}$. In particular, one can construct mixed binary/ternary codes via senary codes by applying the Chinese Gray map (see Example 1). Motivated by this (apart from practical applications such as PSK modulation [4]), in this paper we consider senary simplex codes, and investigate their fundamental properties. We also study their Chinese product type construction.

A *linear code* $\mathcal{C}$, of length n, over $\mathbb{Z}_6$ is an additive subgroup of $\mathbb{Z}_6^n$. An element of $\mathcal{C}$ is called a *codeword of* $\mathcal{C}$ and a *generator matrix* of $\mathcal{C}$ is a matrix whose rows generate $\mathcal{C}$. The *Hamming weight* $w_H(x)$ of a vector x in $\mathbb{Z}_6^n$ is the number of non-zero components. The *Lee weight* $w_L(x)$ of a vector $x = (x_1, x_2, \ldots, x_n)$ is $\sum_{i=1}^n \min\{|x_i|, |6-x_i|\}$. The *Euclidean weight* $w_E(x)$ of a vector x is $\sum_{i=1}^n \min\{x_i^2, (6-x_i)^2\}$. The Euclidean weight is useful in connection with lattice constructions. The *Chinese Euclidean weight* $w_{CE}(x)$ of a vector $x \in \mathbb{Z}_m^n$ is $\sum_{i=1}^n \left\{2 - 2\cos\left(\frac{2\pi x_i}{m}\right)\right\}$. This is useful for $m-PSK$ coding [4]. The Hamming, Lee and Euclidean distances $d_H(x,y)$, $d_L(x,y)$ and $d_E(x,y)$ between two vectors x and y are $w_H(x-y)$, $w_L(x-y)$ and $w_E(x-y)$, respectively. The minimum Hamming, Lee and Euclidean weights, d_H, d_L and d_E, of $\mathcal{C}$ are the smallest Hamming, Lee and Euclidean weights among all non-zero codewords of $\mathcal{C}$, respectively.

The *Chinese Gray map* $\phi: \mathbb{Z}_6^n \to \mathbb{Z}_2^n\mathbb{Z}_3^n$ is the coordinate-wise extension of the function from $\mathbb{Z}_6$ to $\mathbb{Z}_2\mathbb{Z}_3$ defined by $0 \to (0,0), 1 \to (1,1), 2 \to (0,2), 3 \to (1,0), 4 \to (0,1)$ and $5 \to (1,2)$. The inverse map ϕ^{-1} is a ring isomorphism and so is ϕ[6]. The image $\phi(\mathcal{C})$, of a linear code $\mathcal{C}$ over $\mathbb{Z}_6$ of length n by the Chinese Gray map, is a mixed binary/ternary code of length $2n$.

The *dual code* $\mathcal{C}^\perp$ of $\mathcal{C}$ is defined as $\{x \in \mathbb{Z}_6^n \mid x \cdot y = 0 \text{ for all } y \in \mathcal{C}\}$ where $x \cdot y$ is the standard inner product of x and y. $\mathcal{C}$ is *self-orthogonal* if $\mathcal{C} \subseteq \mathcal{C}^\perp$ and $\mathcal{C}$ is *self-dual* if $\mathcal{C} = \mathcal{C}^\perp$.

Two codes are said to be *equivalent* if one can be obtained from the other by permuting the coordinates and (if necessary) changing the signs of certain coordinates. Codes differing by only a permutation of coordinates are called *permutation-equivalent.*

In this paper we define $\mathbb{Z}_6$-simplex codes of type α, β and γ namely, S_k^α, S_k^β and S_k^γ, and determine some of their fundamental parameters. Section 2 contains some preliminaries and notations. Definitions and basic parameters of $\mathbb{Z}_6$-simplex codes of type α, β and γ are given in Section 3. Section 4 investigates their Chinese product type construction.

2 Preliminaries and Notations

Any linear code $\mathcal{C}$ over $\mathbb{Z}_6$ is permutation-equivalent to a code with generator matrix G (the rows of G generate $\mathcal{C}$) of the form

$$G = \begin{bmatrix} I_{k_1} & A_{1,2} & A_{1,3} & A_{1,4} \\ 0 & 2I_{k_2} & 2A_{2,3} & 2A_{2,4} \\ 0 & 0 & 3I_{k_3} & 3A_{3,4} \end{bmatrix}, \tag{1}$$

where the $A_{i,j}$ are matrices with entries 0 or 1 for $i > 1$, and I_k is the identity matrix of order k. Such a code is said to have rank $\{1^{k_1}, 2^{k_2}, 3^{k_3}\}$ or simply rank $\{k_1, k_2, k_3\}$ and $|\mathcal{C}| = 6^{k_1}3^{k_2}2^{k_3}$ [1]. If $k_2 = k_3 = 0$ then the rank of $\mathcal{C}$ is $\{k_1, 0, 0\}$ or simply $k_1 = k$.

To each code $\mathcal{C}$ one can associate two *residue codes* viz $\mathcal{C}_2$ and $\mathcal{C}_3$ defined as

$$\mathcal{C}_2 = \{v \mid v \equiv w \pmod 2, \quad w \in \mathcal{C}\},$$

and

$$\mathcal{C}_3 = \{v \mid v \equiv w \pmod 3, \quad w \in \mathcal{C}\}.$$

Code $\mathcal{C}_2$ is permutation-equivalent to a code with generator matrix of the form

$$\begin{pmatrix} I_{k_1} & A_{1,2} & A_{1,3} & A_{1,4} \\ 0 & 0 & 3I_{k_3} & 3A_{3,4} \end{pmatrix}, \tag{2}$$

where $A_{i,j}$ are binary matrices for $i > 1$. Note that $\mathcal{C}_2$ has dimension $k_1 + k_3$. The ternary code $\mathcal{C}_3$ is permutation-equivalent to a code with generator matrix of the form

$$\begin{pmatrix} I_{k_1} & A_{1,2} & A_{1,3} & A_{1,4} \\ 0 & 2I_{k_2} & 2A_{2,3} & 2A_{2,4} \end{pmatrix}, \tag{3}$$

where $A_{i,j}$ are binary matrices for $i > 1$. Note that $\mathcal{C}_3$ has dimension $k_1 + k_2$.

One can also associate two *torsion codes* with $\mathcal{C}$ viz $\mathcal{C}_2{}^\star$ and $\mathcal{C}_3^\star$ defined as

$$\mathcal{C}_2^\star = \left\{ \frac{c}{3} \mid c = (c_1, \ldots, c_n) \in \mathcal{C} \text{ and } c_i \equiv 0 \pmod 3 \text{ for } 1 \le i \le n \right\}$$

and

$$\mathcal{C}_3^\star = \left\{ \frac{c}{2} \mid c = (c_1, \ldots, c_n) \in \mathcal{C} \text{ and } c_i \equiv 0 \pmod 2 \text{ for } 1 \le i \le n \right\}.$$

If $k_2 = k_3 = 0$ then $C_i = \mathcal{C}_i^\star$ for $i = 2, 3$.

A linear code $\mathcal{C}$ over $\mathbb{Z}_6$ of length n and rank $\{k_1, k_2, k_3\}$ is called an $[n; k_1, k_2, k_3]$ code. If $k_2 = k_3 = 0$, $\mathcal{C}$ is called an $[n, k]$ code. In the case of simplex codes we indeed have $k_2 = k_3 = 0$.

Let $\mathcal{C} : [n; k_1, k_2, k_3]$ be a code over $\mathbb{Z}_6$. For $r_1 \le k_1, r_2 \le k_2, r_1 + r_2 + r_3 \le k_1 + k_2 + k_3$, the *Generalized Hamming Weight* of $\mathcal{C}$ is defined by

$$d_{r_1,r_2,r_3} = \min \{ w_S(\mathcal{D}) \mid \mathcal{D} \text{ is an } [n; r_1, r_2, r_3] \text{ subcode of } \mathcal{C} \},$$

where $w_S(\mathcal{D})$, called *support size* of $\mathcal{D}$, is the number of coordinates in which some codeword of $\mathcal{D}$ has a nonzero entry. The set $\{d_{r_1,r_2,r_3}\}$ is called the *weight hierarchy* of $\mathcal{C}$.

We have the following Lemma connecting the support weight and the Chinese Euclidean weight.

Lemma 1. *Let $\mathcal{D} : [n; r_1, r_2, r_3]$ be a senary linear code then*

$$\sum_{\mathbf{c} \in \mathcal{D}} w_{CE}(\mathbf{c}) = 2^{r_1+r_3+1} \cdot 3^{r_1+r_2} \cdot w_S(\mathcal{D}).$$

Proof. Consider the $(r \times n)$ array of all the codewords in $\mathcal{D}$ (where $r = 6^{r_1} 3^{r_2} 2^{r_3}$). It is easy to see that each column consists of either

1. only zeros
2. 0 and 3 equally often
3. $0, 2$ and 4 equally often
4. $0, 1, 2, 3, 4, 5$ equally often.

Let $n_i, i = 0, 1, 2, 3$ be the number of columns of each type. Then $n_1 + n_2 + n_3 = w_S(\mathcal{D})$. Now applying the standard arguments to evaluate the sum yields the result.

Thus for any linear code $\mathcal{C}$ over $\mathbb{Z}_6$, d_{r_1,r_2,r_3} may also be defined by

$$d_{r_1,r_2,r_3} = \frac{1}{2^{r_1+r_3+1} \cdot 3^{r_1+r_2}} \min \left\{ \sum_{\mathbf{c} \in \mathcal{D}} w_{CE}(\mathbf{c}) \mid \mathcal{D} \text{ is an } [n; r_1, r_2, r_3] \text{ subcode of } \mathcal{C} \right\}.$$

3 Senary Simplex Codes of Type α, β and γ

Let G_k^α be a $k \times 2^k 3^k$ matrix over $\mathbb{Z}_6$ consisting of all possible distinct columns. Inductively, G_k^α may be written as

$$G_k^\alpha = \left[\begin{array}{c|c|c|c|c|c} 00\cdots0 & 11\cdots1 & 22\cdots2 & 33\cdots3 & 44\cdots4 & 55\cdots5 \\ \hline G_{k-1}^\alpha & G_{k-1}^\alpha & G_{k-1}^\alpha & G_{k-1}^\alpha & G_{k-1}^\alpha & G_{k-1}^\alpha \end{array}\right]_{k\times 6^k}$$

with G_1^α =[012345]. The code S_k^α generated by G_k^α has length 6^k and the rank of S_k^α is $\{k, 0, 0\}$.

The following observations are useful to obtain the weight distribution of S_k^α.

Remark 1. If A_{k-1} denotes the $(6^{k-1} \times 6^{k-1})$ array consisting of all codewords in S_{k-1}^α, and if J is the matrix of all $1's$ then the $(6^k \times 6^k)$ array of codewords of S_k^α is given by

$$\begin{bmatrix} A_{k-1} & A_{k-1} & A_{k-1} & A_{k-1} & A_{k-1} & A_{k-1} \\ A_{k-1} & J + A_{k-1} & 2J + A_{k-1} & 3J + A_{k-1} & 4J + A_{k-1} & 5J + A_{k-1} \\ A_{k-1} & 2J + A_{k-1} & 4J + A_{k-1} & A_{k-1} & 2J + A_{k-1} & 4J + A_{k-1} \\ A_{k-1} & 3J + A_{k-1} & A_{k-1} & 3J + A_{k-1} & A_{k-1} & 3J + A_{k-1} \\ A_{k-1} & 4J + A_{k-1} & 2J + A_{k-1} & A_{k-1} & 4J + A_{k-1} & 2J + A_{k-1} \\ A_{k-1} & 5J + A_{k-1} & 4J + A_{k-1} & 3J + A_{k-1} & 2J + A_{k-1} & 1J + A_{k-1} \end{bmatrix}.$$

Remark 2. If $R_1, R_2, ..., R_k$ denote the rows of the matrix G_k^α then $w_H(R_i) = 5 \cdot 6^{k-1}, w_L(R_i) = 9 \cdot 6^{k-1}, w_E(R_i) = 19 \cdot 6^{k-1}$ and $w_{CE}(R_i) = 2 \cdot 6^k$.

It may be observed that each element of $\mathbb{Z}_6$ occurs equally often in every row of G_k^α. Let $\mathbf{c} = (c_1, c_2, \ldots, c_n) \in \mathcal{C}$. For each $j \in \mathbb{Z}_6$ let $\omega_j(\mathbf{c}) = |\{i \mid c_i = j\}|$. We have the following lemma.

Lemma 2. *Let* $\mathbf{c} \in S_k^\alpha, \mathbf{c} \neq 0$.

1. *If for at least one* i*,* c_i *is a unit* $(1$ *or* $5)$ *then* $\forall j \in \mathbb{Z}_6$ $\omega_j = 2^{k-1} \cdot 3^{k-1}$ *in* $\mathbf{c}$.
2. *If* $\forall i,$ $c_i \in \{0, \pm 2\}$ *then* $\forall j \in \{0, \pm 2\}$ $\omega_j = 2^k \cdot 3^{k-1}$ *in* $\mathbf{c}$.
3. *If* $\forall i,$ $c_i \in \{0, 3\}$ *then* $\forall j \in \{0, 3\}$ $\omega_j = 2^{k-1} \cdot 3^k$ *in* $\mathbf{c}$.

Proof. By Remark 1, any $x \in S_{k-1}^\alpha$ gives rise to six codewords of S_k^α:

$$\begin{aligned} y_1 &= \left(x \middle| x \middle| x \middle| x \middle| x \middle| x \right), \\ y_2 &= \left(x \middle| \mathbf{1} + x \middle| \mathbf{2} + x \middle| \mathbf{3} + x \middle| \mathbf{4} + x \middle| \mathbf{5} + x \right), \\ y_3 &= \left(x \middle| \mathbf{2} + x \middle| \mathbf{4} + x \middle| x \middle| \mathbf{2} + x \middle| \mathbf{4} + x \right), \\ y_4 &= \left(x \middle| \mathbf{3} + x \middle| x \middle| \mathbf{3} + x \middle| x \middle| \mathbf{3} + x \right), \\ y_5 &= \left(x \middle| \mathbf{4} + x \middle| \mathbf{2} + x \middle| x \middle| \mathbf{4} + x \middle| \mathbf{2} + x \right), \\ &\text{and} \\ y_6 &= \left(x \middle| \mathbf{5} + x \middle| \mathbf{4} + x \middle| \mathbf{3} + x \middle| \mathbf{2} + x \middle| \mathbf{1} + x \right), \text{ where } \mathbf{i} = (iii...i). \end{aligned}$$

Now the result can be easily proved by induction on k.

Now we recall some known facts about binary and ternary simplex codes. Let $G(S_k)$ (columns consisting of all non-zero binary k-tuples) be a generator matrix for an $[n,k]$ binary simplex code S_k. Then the extended binary simplex code (also known as a type α binary simplex code), $\hat{S_k}$ is generated by the matrix $G(\hat{S_k}) = [\mathbf{0}\ G(S_k)]$. Inductively,

$$G(\hat{S_k}) = \left[\begin{array}{c|c} 00\ldots0 & 11\ldots1 \\ \hline G(\hat{S_{k-1}}) & G(\hat{S_{k-1}}) \end{array}\right] \text{ with } G(\hat{S_1}) = [01]\,. \tag{4}$$

The ternary simplex code of type α is defined inductively by

$$T_k^\alpha = \left[\begin{array}{c|c|c} 00\cdots0 & 11\cdots1 & 22\cdots2 \\ \hline T_{k-1}^\alpha & T_{k-1}^\alpha & T_{k-1}^\alpha \end{array}\right] \text{ with } T_1^\alpha = [012]\,, \tag{5}$$

and the ternary simplex code is defined by the usual generator matrix as

$$T_k^\beta = \left[\begin{array}{c|c} 11\cdots1 & 00\cdots0 \\ \hline T_{k-1}^\alpha & T_{k-1}^\beta \end{array}\right] \text{ with } T_2^\beta = \left[\begin{array}{c|c} 111 & 0 \\ 012 & 1 \end{array}\right].$$

Now we determine the torsion codes of the senary simplex code of type α.

Lemma 3. *The binary (ternary) torsion code of S_k^α is equivalent to 3^k copies of the binary type α simplex code $\big(2^k$ copies of the ternary type α simplex code$\big)$.*

Proof. We will prove the binary case by induction on k. The proof of ternary case is similar and so is omitted. Observe that the binary torsion code of S_k^α is the set of codewords obtained by replacing 3 by 1 in all 2-linear combinations of the rows of the matrix

$$3G_k^\alpha = \left[\begin{array}{c|c|c|c|c|c} 00\cdots0 & 33\cdots3 & 00\cdots0 & 33\cdots3 & 00\cdots0 & 33\cdots3 \\ \hline 3G_{k-1}^\alpha & 3G_{k-1}^\alpha & 3G_{k-1}^\alpha & 3G_{k-1}^\alpha & 3G_{k-1}^\alpha & 3G_{k-1}^\alpha \end{array}\right]. \tag{6}$$

Clearly the result holds for $k = 2$. Assuming that the binary torsion code is equivalent to the 3^{k-1} copies of the extended binary simplex code, we have $\big[3G(\hat{S_{k-1}})\big|\cdots\big|3G(\hat{S_{k-1}})\big]$ in place of $3G_{k-1}^\alpha$ in the above matrix. Now regrouping the columns in the above matrix according to (4) yields the desired result.

As a consequence of Lemmas 2 and 3, one gets the weight distribution of S_k^α.

Theorem 1. *The Hamming, Lee, Euclidean and C-Euclidean weight distributions of S_k^α are*

1. $A_H(0) = 1, A_H(3\cdot6^{k-1}) = (2^k-1), A_H(4\cdot6^{k-1}) = (3^k-1)$, $A_H(5\cdot6^{k-1}) = (2^k-1)(3^k-1)$.
2. $A_L(0) = 1, A_L(8\cdot6^{k-1}) = (3^k-1), A_L(9\cdot6^{k-1}) = 3^k(2^k-1)-1$.
3. $A_E(0) = 1, A_E(27\cdot6^{k-1}) = (2^k-1), A_E(16\cdot6^{k-1}) = (3^k-1)$, $A_E(19\cdot6^{k-1}) = (2^k-1)(3^k-1)$.
4. $A_{CE}(0) = 1, A_{CE}(2\cdot6^k) = 3^k\cdot2^k-1$,
   where $A_H(i)$ $(A_L(i))$ denotes the number of vectors of Hamming (Lee) weight i in S_k^α, and similarly for the Euclidean weights of both types.

Proof. By Lemma 2, each non-zero codeword of S_k^{α} has Hamming weight either $3 \cdot 6^{k-1}$, $4 \cdot 6^{k-1}$, or $5 \cdot 6^{k-1}$ and Lee weight either $8 \cdot 6^{k-1}$ or $9 \cdot 6^{k-1}$. Since the dimension of the binary torsion code is k, there will be $2^k - 1$ codewords of the Hamming weight $3 \cdot 6^{k-1}$, and the dimension of the ternary torsion code is k, so there will be $3^k - 1$ codewords of the Hamming weight $4 \cdot 6^{k-1}$. Hence the number of codewords having Hamming weight $5 \cdot 6^{k-1}$ will be $6^k - \big(3^k + 2^k - 1\big)$. Similar arguments hold for the other weights.

The *symmetrized weight enumerator (swe)* of a senary code $\mathcal{C}$ is defined as

$$swe_{\mathcal{C}}(a,b,c,d) := \sum_{x \in \mathcal{C}} a^{n_0(x)} b^{n_1(x)} c^{n_2(x)} d^{n_3(x)},$$

where $n_i(x)$ denotes the number of j such that $x_j = \pm i$. Let $\bar{S_k^{\alpha}}$ be the punctured code of S_k^{α} obtained by deleting the zero coordinate. Then the swe of $\bar{S_k^{\alpha}}$ is

$$swe_{\bar{S_k^{\alpha}}}(a,b,c,d) = 1 + (2^k-1)d(ad)^{3 \cdot 6^{k-1}-1} + \\ (3^k-1)a^{2 \cdot 6^{k-1}-1}c^{4 \cdot 6^{k-1}} + (2^k-1)(3^k-1)d(ad)^{6^{k-1}-1}(bc)^{2 \cdot 6^{k-1}}.$$

Remark 3. 1. S_k^{α} is an equidistant code with respect to Chinese Euclidean distance whereas the binary (quaternary i.e, over $\mathbb{Z}_4$) simplex code is equidistant with respect to Hamming (Lee) distance.
2. The minimum weights of S_k^{α} are: $d_H = 3 \cdot 6^{k-1}, d_L = 8 \cdot 6^{k-1}, d_E = 16 \cdot 6^{k-1}, d_{CE} = 2 \cdot 6^k$.

Example 1. Consider the $6^4 = 1296$ codewords of the senary code generated by the following generator matrix

$$\begin{array}{c} 11111111 \\ 22220000 \\ 22002200 \\ 20202020 \\ 33334444 \\ 33443344 \\ 34343434 \end{array}.$$

Using the Chinese Gray map results in a mixed code with 8 binary and 8 ternary coordinates, which gives $N(8,8,4) \geq 1296$, while the ternary code of length 8, dimension 4 and distance 4 is optimal [3].

Let Λ_k be the $k \times 3^k \cdot (2^k - 1)$ matrix defined inductively by $\Lambda_1 = [135]$ and

$$\Lambda_k = \left[\begin{array}{c|c|c|c|c|c} 0 \ldots 0 & 1 \cdots 1 & 2 \cdots 2 & 3 \cdots 3 & 4 \cdots 4 & 5 \cdots 5 \\ \hline \Lambda_{k-1} & G_{k-1}^{\alpha} & \Lambda_{k-1} & G_{k-1}^{\alpha} & \Lambda_{k-1} & G_{k-1}^{\alpha} \end{array} \right],$$

for $k \geq 2$; and let μ_k be the $k \times 2^{k-1} \cdot (3^k - 1)$ matrix defined inductively by $\mu_1 = [12]$ and

$$\mu_k = \left[\begin{array}{c|c|c|c} 0 \cdots 0 & 1 \cdots 1 & 2 \cdots 2 & 3 \cdots 3 \\ \hline \mu_{k-1} & G_{k-1}^{\alpha} & G_{k-1}^{\alpha} & \mu_{k-1} \end{array} \right],$$

for $k \geq 2$, where G^{α}_{k-1} is the generator matrix of S^{α}_{k-1}.

Now let G^{β}_{k} be the $k \times \frac{(2^k-1)(3^k-1)}{2}$ matrix defined inductively by

$$G^{\beta}_{2} = \left[\begin{array}{c|c|c|c} 111111 & 0 & 222 & 33 \\ \hline 012345 & 1 & 135 & 12 \end{array}\right],$$

and for $k > 2$

$$G^{\beta}_{k} = \left[\begin{array}{c|c|c|c} 11\cdots1 & 00\cdots0 & 22\cdots2 & 33\cdots3 \\ \hline G^{\alpha}_{k-1} & G^{\beta}_{k-1} & \Lambda_{k-1} & \mu_{k-1} \end{array}\right],$$

where G^{α}_{k-1} is the generator matrix of S^{α}_{k-1}. Note that G^{β}_{k} is obtained from G^{α}_{k} by deleting $\frac{(2^k+1)(3^k-1)+2^{k+1}}{2}$ columns. By induction it is easy to verify that no two columns of G^{β}_{k} are multiples of each other. Let S^{β}_{k} be the code generated by G^{β}_{k}. Note that S^{β}_{k} is a $\left[\frac{(2^k-1)(3^k-1)}{2}, k\right]$ code. To determine the weight distributions of S^{β}_{k} we first make some observations.

The proof of the following proposition is similar to that of Proposition 2.

Proposition 1. *Each row of G^{β}_{k} contains 6^{k-1} units and*
$\omega_2 + \omega_4 = 3^{k-1}(2^{k-1}-1), \omega_3 = 2^{k-2}(3^{k-1}-1), \omega_0 = \frac{(2^{k-1}-1)(3^{k-1}-1)}{2}$.

Remark 4. Each row of G^{β}_{k} has Hamming weight $\left(3^{k-1}\cdot(2^k-1)+2^{k-2}\cdot(3^{k-1}-1)\right)$, Lee weight $\left(2\cdot 3^{k-1}(3\cdot 2^{k-2}-1)+3\cdot 2^{k-2}(3^{k-1}-1)\right)$, Euclidean weight $\left(3^{k-1}(19\cdot 2^{k-2}-4)-9\cdot 2^{k-2}\right)$, and Chinese Euclidean weight $6^k-2^k-3^k$.

The proof of the following lemma is similar to the proof of Lemma 2.

Lemma 4. *Let $\mathbf{c} \in S^{\beta}_{k}, \mathbf{c} \neq 0$.*

1. *If for at least one i, c_i is a unit then $\forall j \in \mathbb{Z}_6$ $\omega_1+\omega_5 = 6^{k-1}$,* $\omega_2+\omega_4 = 3^{k-1}(2^{k-1}-1), \omega_3 = 2^{k-2}(3^{k-1}-1), \omega_0 = \frac{(2^{k-1}-1)(3^{k-1}-1)}{2}$ *in* $\mathbf{c}$.
2. *If $\forall i, c_i \in \{0, \pm 2\}$ then $\forall j \in \{0, \pm 2\}$ $\omega_2+\omega_4 = 3^{k-1}(2^k-1), \omega_0 = \frac{(2^k-1)(3^{k-1}-1)}{2}$ in* $\mathbf{c}$.
3. *If $\forall i$, $c_i \in \{0,3\}$ then $\forall j \in \{0,3\}$ $\omega_3 = 2^{k-2}(3^k-1), \omega_0 = \frac{(2^{k-1}-1)(3^k-1)}{2}$ in* $\mathbf{c}$.

The proof of the following lemma is similar to that of Lemma 3 and is omitted.

Lemma 5. *The binary (ternary) torsion code of S^{β}_{k} is equivalent to $\frac{(3^k-1)}{2}$ copies of the binary simplex code $\left((2^k-1)\right.$ copies of the ternary simplex code$\left.\right)$.*

The proof of the following theorem is similar to that of Theorem 1 and is omitted.

Theorem 2. *The Hamming, Lee weight, Euclidean and C-Euclidean weight distributions of S^{β}_{k} are:*

1. $A_H(0) = 1, A_H\left(2^{k-2}\cdot(3^k-1)\right) = (2^k-1), A_H\left(3^{k-1}\cdot(2^k-1)\right) = (3^k-1)$, $A_H\left(3^{k-1}\cdot(2^k-1)+2^{k-2}\cdot(3^{k-1}-1)\right) = (2^k-1)(3^k-1)$.

2. $A_L(0) = 1, A_L\left(3 \cdot 2^{k-2}(3^k - 1)\right) = (2^k - 1), A_L\left(2 \cdot 3^{k-1}(2^k - 1)\right) = (3^k - 1),$ $A_L\left(2 \cdot 3^{k-1}(3 \cdot 2^{k-2} - 1) + 3 \cdot 2^{k-2}(3^{k-1} - 1)\right) = (2^k - 1)(3^k - 1).$

3. $A_E(0) = 1, A_E\left(9 \cdot 2^{k-2}(3^k - 1)\right) = (2^k - 1), A_E\left(4 \cdot 3^{k-1}(2^k - 1)\right) = (3^k - 1), A_E\left(3^{k-1}(19 \cdot 2^{k-2} - 4) - 9 \cdot 2^{k-2}\right) = (2^k - 1)(3^k - 1).$

4. $A_{CE}(0) = 1, A_{CE}(6^k - 2^k) = (2^k - 1), A_{CE}(6^k - 3^k) = (3^k - 1),$ $A_{CE}(6^k - 2^k - 3^k) = (2^k - 1)(3^k - 1),$ *where $A_H(i)$ $(A_L(i))$ denotes the number of vectors of Hamming (Lee) weight i in S_k^α, and similarly for the Euclidean weights of both types.*

Remark 5. 1. The swe of S_k^β is given as

$$swe(a,b,c,d) = 1 + 3^{-k}p(k)a^{n(k-1)+p(k-1)}d^{2^{-1}q(k)} + 2^{-k+1}q(k)a^{n(k-1)}\left\{a^{q(k-1)}c^{3^{-1}p(k)} + 3^{-k}p(k)b^{6^{k-1}}c^{p(k-1)}d^{q(k-1)}\right\}.$$

where $n(k) = \frac{(2^k-1)(3^k-1)}{2}$, $p(k) = 3^k(2^k - 1)$ and $q(k) = 2^{k-1}(3^k - 1)$.

2. The minimum weights of S_k^β are: $d_H = 2^{k-2}(3^k - 1), d_L = 2 \cdot 3^{k-1}(2^k - 1), d_E = 4 \cdot 3^{k-1}(2^k - 1), d_{CE} = 6^k - 2^k - 3^k$.

Let G_k^γ be the $k \times 2^{k-1}(3^k - 2^k)$ matrix defined inductively by

$$G_2^\gamma = \left[\begin{array}{c|c|c|c|c} 111111 & 0 & 2 & 3 & 4 \\ \hline 012345 & 1 & 1 & 1 & 1 \end{array}\right],$$

and for $k > 2$

$$G_k^\gamma = \left[\begin{array}{c|c|c|c|c} 11\cdots1 & 00\cdots0 & 22\cdots2 & 33\cdots3 & 44\cdots4 \\ \hline G_{k-1}^\alpha & G_{k-1}^\gamma & G_{k-1}^\gamma & G_{k-1}^\gamma & G_{k-1}^\gamma \end{array}\right],$$

where G_{k-1}^α is the generator matrix of S_{k-1}^α. Note that G_k^γ is obtained from G_k^α by deleting $2^{k-1}(2^k + 3^k)$ columns. By induction it is easy to verify that no two columns of G_k^γ are multiples of each other. Let S_k^γ be the code generated by G_k^γ. Note that S_k^γ is a $\left[2^{k-1}(3^k - 2^k), k\right]$ code.

Proposition 2. *Each row of G_k^γ contains $6^{(k-1)}$ units and* $\omega_0 = \omega_2 = \omega_3 = \omega_4 = 2^{k-2}(3^{k-1} - 2^{k-1})$.

Proof. Clearly the assertion holds for the first row. Assume that the result holds for each row of G_{k-1}^γ. Then the number of units in each row of G_{k-1}^γ is $6^{(k-2)}$. By Lemma 2, the number of units in any row of G_{k-1}^α is $2^{k-1} \cdot 3^{k-2}$. Hence the total number of units in any row of G_k^γ will be $2^{k-1} \cdot 3^{k-2} + 4 \cdot 2^{k-2} \cdot 3^{k-2} = 2^{k-1} \cdot 3^{k-1}$. A similar argument holds for the number of 0's, 2's, 3's and 4's.

Remark 6. Each row of G_k^γ has Hamming weight $3 \cdot 2^{k-2}\left[5 \cdot 3^{k-2} - 2^{k-1}\right]$, Lee weight $2^{k-2}\left[3^{k+1} - 7 \cdot 2^{k-1}\right]$, Euclidean weight $2^{k-2}\left[19 \cdot 3^{k-1} - 17 \cdot 2^{k-1}\right]$, and Chinese Euclidean weight $6^k - 5 \cdot 4^{k-1}$.

The various weight distributions of S_k^γ can be obtained using arguments similar to other simplex codes. To save the space we omit them.

The weight hierarchy of S_k^α is given by the following theorem.

Theorem 3. *The weight hierarchy of S_k^α is given by*

$$d_{r_1,r_2,r_3}(S_k^\alpha) = 6^k - 3^{k-r_1-r_2} \cdot 2^{k-r_1-r_3}.$$

Proof. By Remark 3 and the definition of d_{r_1,r_2,r_3} after the Lemma 1.

4 Chinese Product Type Construction

The Chinese remainder theorem (CRT) plays an important role in the study of codes over $\mathbb{Z}_{2k}$ [4, 6]. In particular, given binary and ternary linear codes of length n and dimension k, one can construct a senary code (over $\mathbb{Z}_6$) of length n using CRT. The following theorem is from [4, 6].

Theorem 4. *[4, 6] If B and T are linear codes of length n over $GF(2)$ and $GF(3)$, respectively, then the set $CRT(B,T) = \left\{\phi^{-1}(\mathbf{c}_b, \mathbf{c}_t) \mid \mathbf{c}_b \in B, \mathbf{c}_t \in T\right\}$ is a linear code of length n over $\mathbb{Z}_6$. Moreover if B and T are self-orthogonal then $CRT(B,T)$ is also self-orthogonal.*

If generator matrices of B, T and $CRT(B,T)$ are $G(B), G(T)$ and $G(CRT(B,T))$, respectively, then we have $\phi(G(CRT(B,T))) = [G(B)|G(T)]$, where ϕ is the Chinese Gray map. If the codes B and T are of different lengths, say, n_1 and n_2 then it seems that no non-trivial method is known to construct a code over $\mathbb{Z}_6$ from these codes. In the trivial case of course one can add extra zero columns to the generator matrix of the code of shorter length and then use Theorem 4. Here we present a new construction of a generator matrix of senary code from codes of different lengths.

Let $G(B) = [x_1 x_2 \ldots x_{n_1}]$ and $G(T) = [y_1 y_2 \ldots y_{n_2}]$ where x_i, y_i are the corresponding columns. Now form the matrix $G(B) \star G(T)$ consisting of the $n_1 n_2$ pairs of total $2n_1 n_2$ columns $\{x_i y_1 x_i y_2 \ldots x_i y_{n_2}\}_{i=1}^{n_1}$. These pairs of columns give a generator matrix of length $n_1 n_2$ (the product of the lengths of the binary and ternary codes) over $\mathbb{Z}_6$ using the inverse Chinese Gray map. In particular, if $n_1 = n_2 = n$ then we get a code of length n^2. Note that if we use the Theorem 4 to construct a generator matrix for the case of $n_1 = n_2 = n$, we obtain a code of length n with generator matrix $[x_1 y_1 x_2 y_2 \ldots x_n y_n]$. In this case, the resulting code will be self orthogonal if the corresponding binary and ternary codes are self orthogonal [6]. Similarly it is easy to see that

Lemma 6. *The senary codes constructed by $G(B) * G(T)$ will be self orthogonal if the corresponding codes B and T are self orthogonal.*

The next two results show that self-orthogonal simplex codes of type α and β can be obtained from the construction $G(B) * G(T)$.

Theorem 5. *The codes S_k^α and S_k^β can be obtained via the construction $G(B) * G(T)$.*

Proof. We will only prove the result for S_k^α, since the other case is similar. If we apply the Chinese Gray map to the generator matrix G_k^α, we see that it is equivalent to the matrix $G(\hat{S_k}) \star T_k^\alpha$, where T_k^α is defined in (5).

Theorem 6. *The codes S_k^α $(k \geq 3)$ and S_k^β $(k \geq 2)$ are self orthogonal.*

Proof. The result follows from Lemma 6 and Theorem 5. It can also be proved by induction on k since the rows of the generator matrices are pairwise orthogonal and each of the rows has Euclidean weight a multiple of 12 [1].

Remark 7. The code S_k^γ is not self-orthogonal as the Euclidean weights of the rows of G_k^γ are not a multiple of 12.

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