On Some Quaternary Self-Orthogonal Codes[1]

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Abstract

This paper studies families of self-orthogonal codes over $\mathbb{Z}_4$. We show that the simplex codes (Type α and Type β) are self-orthogonal. We partially answer the question of $\mathbb{Z}_4$-linearity for the codes from projective planes of even order. A new family of self-orthogonal codes over $\mathbb{Z}_4$ is constructed via projective planes of odd order. Properties such as shadow codes, self-orthogonality, weight distribution, etc. are studied. Finally, some self-orthogonal codes constructed from twistulant matrices are presented.

Keywords: Self-orthogonal codes, codes over rings, projective planes, simplex codes.

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1 Introduction

There has been considerable interest and research in codes over finite rings in recent years, In particular codes over $\mathbb{Z}_4$ have been widely studied [1, 2, 8, 9, 10, 20, 24, 25]. Self-orthogonal and self-dual codes have received much attention. An excellent survey of self-dual codes is given by Rains and Sloane [24]. In this paper we consider several families of self-orthogonal codes over $\mathbb{Z}_4$ and investigate their properties.

A *linear code* $\mathcal{C}$, of length n, over $\mathbb{Z}_4$ is an additive subgroup of $\mathbb{Z}_4^n$. An element of $\mathcal{C}$ is called a *codeword of* $\mathcal{C}$ and a *generator matrix* of $\mathcal{C}$ is a matrix whose rows generate $\mathcal{C}$. The *Hamming weight* $w_H(x)$ of a vector x in $\mathbb{Z}_4^n$ is the number of non-zero components. The *Lee weight* $w_L(x)$ of a vector $x = (x_1, x_2, \ldots, x_n)$ is $\sum_{i=1}^n \min\{|x_i|, |4 - x_i|\}$. The *Euclidean weight* $w_E(x)$ of a vector x is $\sum_{i=1}^n \min\{x_i^2, (4 - x_i)^2\}$. The Euclidean weight is useful in connection with lattice constructions. The Hamming, Lee and Euclidean distances $d_H(x, y)$, $d_L(x, y)$ and $d_E(x, y)$ between two vectors x and y are $w_H(x - y)$, $w_L(x - y)$ and $w_E(x - y)$, respectively. The minimum Hamming, Lee and Euclidean weights, d_H, d_L and d_E, of $\mathcal{C}$ are the smallest Hamming, Lee and Euclidean weights respectively amongst all non-zero codewords of $\mathcal{C}$.

The *Gray map* $\phi : \mathbb{Z}_4^n \to \mathbb{Z}_2^{2n}$ is the coordinate-wise extension of the function from $\mathbb{Z}_4$ to $\mathbb{Z}_2^2$ defined by $0 \to (0, 0), 1 \to (0, 1), 2 \to (1, 1), 3 \to (1, 0)$. The image $\phi(\mathcal{C})$, of a linear code $\mathcal{C}$ over $\mathbb{Z}_4$ of length n by the Gray map, is a binary code of length $2n$.

The *dual code* $\mathcal{C}^\perp$ of $\mathcal{C}$ is defined as $\{x \in \mathbb{Z}_4^n \mid x \cdot y = 0 \text{ for all } y \in \mathcal{C}\}$ where $x \cdot y$ is the standard inner product of x and y. $\mathcal{C}$ is *self-orthogonal* if $\mathcal{C} \subseteq \mathcal{C}^\perp$ and $\mathcal{C}$ is *self-dual* if $\mathcal{C} = \mathcal{C}^\perp$.

If C is a binary self-dual code, it is said to be Type II, or doubly-even, if all of the Hamming weights are divisible by four, and is said to be Type I otherwise.

Two codes are said to be *equivalent* if one can be obtained from the other by permuting the coordinates and (if necessary) changing the signs of certain coordinates. Codes differing by only a permutation of coordinates are called *permutation-equivalent*.

In this paper we give a simple characterization of self-orthogonal codes over $\mathbb{Z}_4$ and we show that $\mathbb{Z}_4$-simplex codes of type α and β, namely S_k^α and S_k^β, are self-orthogonal. We also construct families of self-orthogonal and self-dual codes over $\mathbb{Z}_4$ via the projective planes of odd order. Section 2 contains some preliminaries and notations. The relationship between quaternary codes and projective planes is given in Section 3, while Section 4 considers projective planes and quantum codes. Finally Section 5 presents some self-orthogonal codes constructed from twistulant matrices.

2 Preliminaries and Notation

Any linear code $\mathcal{C}$ over $\mathbb{Z}_4$ is permutation-equivalent to a code with generator matrix G of the form

$$G = \begin{pmatrix} I_{k_0} & A & B_1 + 2B_2 \\ 0 & 2I_{k_1} & 2C \end{pmatrix}, \tag{1}$$

where A, B_1, B_2 and C are matrices with entries 0 or 1 and I_k is the identity matrix of order k. One can associate two binary linear codes with $\mathcal{C}$, the *residue code* $\mathcal{C}^{(1)} = \{\mathbf{c} \pmod 2 \mid \mathbf{c} \in \mathcal{C}\}$ and the *torsion code* $\mathcal{C}^{(2)} = \{\mathbf{c} \in \mathbb{Z}_2^n \mid 2\mathbf{c} \in \mathcal{C}\}$. If $k_1 = 0$ then $\mathcal{C}^{(1)} = \mathcal{C}^{(2)}$. For details and further references see [24, 25]. The following theorem gives the relationships between these codes when they are self-orthogonal.

Theorem 1 *[24]*

1. *Let $\mathcal{C}$ be a linear self-orthogonal code over $\mathbb{Z}_4$ then its residue code $\mathcal{C}^{(1)}$ is a self-orthogonal doubly even binary code and $\mathcal{C}^{(1)} \subset \mathcal{C}^{(2)} \subset \mathcal{C}^{(1)^\perp}$, and if $\mathcal{C}$ is self-dual then $\mathcal{C}^{(2)} = \mathcal{C}^{(1)^\perp}$.*

2. *If $\mathcal{C}_A$ and $\mathcal{C}_B$ are binary codes with $\mathcal{C}_A \subset \mathcal{C}_B$ then there is a code $\mathcal{C}$ over $\mathbb{Z}_4$ with $\mathcal{C}^{(1)} = \mathcal{C}_A$ and $\mathcal{C}^{(2)} = \mathcal{C}_B$. If in addition $\mathcal{C}_A$ is self-orthogonal and doubly even and $\mathcal{C}_B \subset \mathcal{C}_A^\perp$ then there is a self-orthogonal code $\mathcal{C}$ over $\mathbb{Z}_4$ with $\mathcal{C}^{(1)} = \mathcal{C}_A$ and $\mathcal{C}^{(2)} = \mathcal{C}_B$. Furthermore if $\mathcal{C}_B = \mathcal{C}_A^\perp$ then $\mathcal{C}$ is self-dual.*

A vector $\mathbf{v}$ is a 2-*linear combination* of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ if $\mathbf{v} = \lambda_1\mathbf{v}_1 + \ldots + \lambda_k\mathbf{v}_k$ with $\lambda_i \in \mathbb{Z}_2$ for $1 \le i \le k$. A subset $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ of $\mathcal{C}$ is called a 2-*basis* for $\mathcal{C}$ if for each $i = 1, 2, ..., k-1$, $2\mathbf{v}_i$ is a $2-$linear combination of $\mathbf{v}_{i+1}, ..., \mathbf{v}_k$, $2\mathbf{v}_k = 0$, $\mathcal{C}$ is the 2-linear span of S and S is 2-linearly independent [2]. The number of elements in a 2-basis for $\mathcal{C}$ is called the 2-*dimension* of $\mathcal{C}$. It is easy to verify that the rows of the matrix

$$\mathcal{B} = \begin{pmatrix} I_{k_0} & A & B_1 + 2B_2 \\ 2I_{k_0} & 2A & 2B_1 \\ 0 & 2I_{k_1} & 2C \end{pmatrix}, \tag{2}$$

form a 2-basis for the code $\mathcal{C}$ generated by G given in (1).

A linear code $\mathcal{C}$ over $\mathbb{Z}_4$ (over $\mathbb{Z}_2$) of length n, 2-dimension k, minimum distance d_H, d_L and d_E is called an $[n, k, d_H, d_L, d_E]$ ($[n, k, d_H]$) or simply an $[n, k]$ code.

Let $\mathcal{C}$ be a self-orthogonal code over $\mathbb{Z}_4$ and let $\mathcal{C}_0$ be the subcode with Euclidean weights divisible by 8. Then the *Shadow* S of $\mathcal{C}$ is defined (see [9, 24]) by

$$S = \begin{cases} \mathcal{C}_0^\perp \backslash \mathcal{C}_0 & \text{if } \mathcal{C} \neq \mathcal{C}_0, \\ \mathcal{C}^\perp & \text{if } \mathcal{C} = \mathcal{C}_0. \end{cases}$$

If $\mathcal{C}$ is a linear code over $\mathbb{Z}_4$ then $\phi(\mathcal{C})$ will denote the image of $\mathcal{C}$ under the Gray map. $\mathcal{C}$ is called $\mathbb{Z}_2$-*Linear* if $\phi(\mathcal{C})$ is a binary linear code. A binary code is said to be $\mathbb{Z}_4$-*linear* if it is equivalent to $\phi(\mathcal{C})$ for some linear code $\mathcal{C}$ over $\mathbb{Z}_4$. A necessary and sufficient condition for $\mathbb{Z}_4$-linearity ($\mathbb{Z}_2$-linearity) is given by the following Theorem.

Theorem 2 Hammons et al *[20]*

1. *A binary linear code C of even length is $\mathbb{Z}_4$-linear if and only if its coordinates can be permuted so that*

   (3) $$\mathbf{u}, \mathbf{v} \in C \Rightarrow (\mathbf{u} + \sigma(\mathbf{u})) \star (\mathbf{v} + \sigma(\mathbf{v})) \in C,$$

   where σ is the swap map that interchanges the left and the right halves of a vector and $\star$ denotes the componentwise product of two vectors.

2. *For each $a \in \mathbb{Z}_4$, let $\bar{a}$ be the reduction of a modulo 2 and let $\mathcal{C}$ be a linear code over $\mathbb{Z}_4$, then $\mathcal{C}$ is $\mathbb{Z}_2$-linear if and only if $\mathbf{c} = (c_1, \ldots, c_n)$ and $\mathbf{c}' = (c'_1, \ldots, c'_n) \in \mathcal{C}$ implies $2\bar{\mathbf{c}} \star \bar{\mathbf{c}}' = (2\bar{c_1}\bar{c_1}', \ldots, 2\bar{c_n}\bar{c_n}') \in \mathcal{C}$.*

The following Lemma gives a simple characterization of the self-orthogonality of codes over $\mathbb{Z}_4$.

Lemma 1 *A linear code $\mathcal{C}$ over $\mathbb{Z}_4$ is self-orthogonal if and only if the number of units in the rows of each generator matrix of $\mathcal{C}$ is a multiple of 4 i.e, $\omega_1 + \omega_3 \equiv 0 \pmod 4$, and every pair of rows is orthogonal.*

Proof. The proof is simple and so is omitted. □

Remark 1 *The above lemma also holds when the generator matrix of the code $\mathcal{C}$ is in 2-basis form. One can also give the characterization in terms of the Euclidean weights.*

Quaternary simplex codes of type α and β have been recently studied in [2]. The following theorem follows by induction on k and Lemma 1.

Theorem 3 *The simplex codes S_k^α $(k \geq 2)$ and S_k^β $(k \geq 2)$ are self-orthogonal.*

2.1 Quasi-Twisted Codes

[16, 17]. The class of quasi-twisted (QT) codes was first introduced in [21] as a generalization of quasi-cyclic (QC) codes [16, 17]. A Z_4 code is called quasi-twisted if a negacyclic[1] shift

[1] A negacyclic shift of an m-tuple $(x_0, x_1, \ldots, x_{m-1})$ is the m-tuple $(\eta x_{m-1}, x_0, \ldots, x_{m-2})$ where η is a unit in Z_4, i.e., $\eta = 1$ or 3.

of a codeword by p positions results in another codeword. Many QT codes codes can be constructed from $m \times m$ twistulant matrices (with a suitable permutation of coordinates). In this case, the generator matrix, G, can be represented as

$$G = [B_1, B_2, \ldots, B_p] \tag{4}$$

where the B_i are $m \times m$ *twistulant* matrices of the form

$$B = \begin{bmatrix} b_0 & b_1 & b_2 & \cdots & b_{m-2} & b_{m-1} \\ \eta b_{m-1} & b_0 & b_1 & \cdots & b_{m-3} & b_{m-2} \\ \eta b_{m-2} & \eta b_{m-1} & b_0 & b_{m-4} & \cdots & b_{m-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \eta b_1 & \eta b_2 & \eta b_3 & \cdots & \eta b_{m-1} & b_0 \end{bmatrix} \tag{5}$$

and $\eta = 1$ or 3. If $\eta = 1$, the code is QC. It has been shown that QT self-dual codes exist only if $\eta = 3$ [18]. Thus in this paper we consider only this value of η.

3 Projective Planes and Quaternary Codes

There have been various constructions of self-dual and self-orthogonal binary codes from projective planes of even order q [24]. These codes were useful in proving the non-existence of the projective plane of order 10. Recently codes over finite rings have been constructed from projective planes [10]. In [14], Glynn constructed binary codes from a projective plane of odd order. In this section we shall associate a code over $\mathbb{Z}_4$ with a certain projective plane of odd order. We also consider the $\mathbb{Z}_4$-linearity of the codes obtained from the projective planes of even order.

Let π be a finite projective plane of order q i.e, a symmetric $2-(q^2+q+1, q+1, 1)$ design. Thus every pair of points determines a unique line and every line contains $q+1$ points. Let $\mathcal{P}$ and $\mathcal{L}$ denote the sets of points and lines, respectively, of π. Then $|\mathcal{P}| = |\mathcal{L}| = q^2+q+1$ and $|\mathcal{P} \cup \mathcal{L}| = 2(q^2+q+1)$. If $q \equiv 2 \pmod 4$ then the incidence matrix of π generates a binary code $\mathcal{C}_q$ with parameters $\left[q^2+q+1, \frac{(q^2+q+2)}{2}, q+1\right]$ which can be extended to a Type II binary self-dual code $\hat{\mathcal{C}}_q$. For $q = 2$ this Type II code is the $[8, 4, 4]$ extended Hamming code which is $\mathbb{Z}_4$-linear under the Gray map [20]. The corresponding code over $\mathbb{Z}_4$ is an $[4, 4, 2, 4]$ Type α constant Lee weight code generated by the matrix

$$G = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}.$$

The corresponding set of lines is

$$\begin{array}{lll} 0202 & 0123 & 1313 \\ 0123 & 1032 & 3012 \\ 1032 & 2103 & 0202 \\ 2103 & 3333 & 0123 \\ 3333 & 1313 & 1032 \\ 1313 & 3012 & 2103 \\ 3012 & 0202 & 3333 \end{array}.$$

Motivated by this we define a code over $\mathbb{Z}_4$ from a projective plane π with the help of pairings of points as follows. We construct a generator matrix of a code $\mathcal{C}(\pi, \mathcal{P}, \mathbb{Z}_4)$ with coordinates corresponding to pairs of points (P, P') in the plane and lines (l) corresponding to rows. The point O is paired with the point at infinity so that $\infty = O'$ and the lines contain the point at infinity. The value on (P, P') is

$$= \begin{cases} 0 & \text{if } P \text{ and } P' \notin l \\ 1 & \text{if } P \in l, P' \notin l \\ 2 & \text{if } P \in l, P' \in l \\ 3 & \text{if } P \notin l, P' \in l \end{cases}.$$

At this point the following question arises.

Question 1 *For what planes of order q is $\hat{\mathcal{C}}_q$ $\mathbb{Z}_4$-linear ?*

The answer to this question in general seems difficult. However we give here a partial answer.

Definition 1 *Given a line $l \in \mathcal{L}$ of π let $\gamma(l) := \{P, P' \mid P \in l \text{and } \mathrm{P}' \notin \mathrm{l}\} \cup \{\mathrm{P}, \mathrm{P}' \mid \mathrm{P} \notin \text{ land } \mathrm{P}' \in \mathrm{l}\}$. Note that $\{O, \infty\} \subseteq \gamma(l) \iff 0 \notin l$.*

We assume that $\phi(C(\pi, \mathcal{P}, \mathbb{Z}_4))$ is contained in $\hat{\mathcal{C}}$. Using condition (ii) of Theorem 2 for distinct pairs of generators $C(\pi, \mathcal{P}, \mathbb{Z}_4)$ coming from two lines in $\mathcal{L}$, we obtain the following result.

Lemma 2 *For any two distinct lines l and m of π, $\gamma(l) \cup \gamma(m)$ is a codeword under the Gray map of $\hat{\mathcal{C}}_q$, or equivalently, the function with the value of 2 on each pair in this set of points is a codeword of $C(\pi, \mathcal{P}, \mathbb{Z}_4)$.*

Let $l * m := \gamma(l) \cup \gamma(m)$, let l be a fixed line not passing through O and let X be any point of l. We consider the sum $S(l, X)$ (in $\hat{\mathcal{C}}_q$, or modulo 2), of $l * m$, where m varies over the q lines of π passing through X, but not l. Then in $S(l, X)$ we have $\{0, \infty\}$ repeated $q - 1$ times, which is odd and therefore it appears again. Further, every pair in $\gamma(l)$ that is not $(0, \infty)$ occurs precisely once in the sum. Since we are assuming that condition (ii) holds we have shown the following.

Lemma 3 *For all lines l not through O, the word corresponding to $\gamma(l)$ is a word in $\hat{\mathcal{C}}_q$.*

Similarly, we consider any line l through O, and look at the sum $T(l)$ of $l * m$, where m is varied over all q lines through O, but not l. Then the sum modulo 2 is $\gamma(l)$. Hence we obtain the above lemma in the case of lines through O. This result can be obtained in a more direct way using condition (ii) of Theorem 2 when $u = v$.

Now a codeword of $\hat{\mathcal{C}}_q$ has the property that any line (union ∞) intersects the word in an even number of points. Thus it follows that for any line l not through O, the lines not through O intersect $\gamma^* := \gamma(l) \setminus \{0, \infty\}$ in an odd number of points, while the lines through O intersect it in an even number of points. Similarly, for any line l on O, the set of points $\gamma(l)$ has the property that any line intersects it in an even number of points. These conditions are quite strong and lead to severe restrictions on the types of pairings of the points of the plane that are possible.

Suppose now that l is a line of π not through O. Then l can contain at most one pair (P, P'). This is because if it does contain such a pair, then each of the $q-1$ lines through P, but not l or PO, intersects $\gamma^*(l)$ in an odd number of points, i.e. at least one further point. But there are now at most $q-1$ points of $\gamma^*(l)$ not on l, and so there can be no further pairs on l, and also each line through P, or P', contains precisely one of these points of $\gamma^*(l)$.

Next, consider a line l of π containing O, and further, suppose that it contains a point of a pair of $\mathcal{P}$, but not both points of the pair. If that point is P, we consider the $q-1$ lines through P, but not the line PP' or l. Since P is in $\gamma(l)$ we see that each of these $q-1$ lines contains a further point of $\gamma(l)$, but since the number of these further points is bounded by $q-1$, we see that there are no pairs of $\mathcal{P}$ completely contained in l, and that $l \setminus \{O\} \subseteq \gamma(l)$.

From the above remarks we have the following.

Lemma 4 *There are two types of pairings $\mathcal{P}$ of the plane π that are possible:*

1. *all pairs lie on lines through O, or*
2. *all pairs lie on lines through O, except for two special lines a, b through O, for which the pairs have one point on a, and one point on b.*

Denote these pairings as Type T1, or Type T2, respectively. Consider a Type T1 pairing, then we have the following.

Lemma 5 *For each line l not through O, the set $\gamma(l) = l \cup H(l) \cup \{\infty\}$, where $H(l)$ is a hyperoval containing O, but disjoint from l.*

Proof. $l \cup \{\infty\} \subset \gamma(l)$, but both $l \cup \{\infty\}$ and $\gamma(l)$ are words of $\hat{\mathcal{C}}_q$. Thus $\gamma(l) \setminus (l \cup \{\infty\})$ is a word of $\hat{\mathcal{C}}_q$ of weight $q+2$, which must be a hyperoval since the minimal words of weight $q+2$ of this code are known to correspond to lines or hyperovals. However, this remaining word does not contain ∞, and so must be a hyperoval.

□

Note that for any line l containing O for a Type T1 pairing, $\gamma(l)$ is the all-zero (or empty) codeword. If we dualize the Type T1 pairing to pairs of lines we can use a construction of Glynn [15] to get a symmetric Hadamard matrix. In this case there are examples with any translation plane of even order which has a dual hyperoval that contains the translation line. There are connections also to Kantor's work, see [4, 23].

Notwithstanding the above connections we can now show that in only the case $q = 2$ can we get a Type T1 pairing that produces a $\mathbb{Z}_4$ code that is also $\mathbb{Z}_2$ linear. In fact, we can now bound the size of q for a Type T1 pairing that satisfies this property.

Going back to the condition that for any two lines l and m of π, $l * m$ is a word of $\hat{\mathcal{C}}_q$, we first choose l to be any line not through O. Then choose m to be any line external to $H(l)$, but not l. This is possible if $q > 2$. Then we see that $l * m$ only contains 4 points: $X := l \cap m$, X', O, and $O' = \infty$. This cannot be a word of $\hat{\mathcal{C}}_q$, unless $q = 2$.

Similarly, in the case of a Type T2 pairing of points of the plane, we can show that the $\mathbb{Z}_4$ code is $\mathbb{Z}_2$ linear only in the case of $q = 2$. Here is an outline of the proof which is brief since it is almost identical to the Type T1 case.

First we show that for all lines l not through O, $\gamma(l)$ is the sum modulo 2 of the line l, a hyperoval containing O, and the point ∞. There are q lines for which the hyperoval is a chord of its corresponding line, and $q^2 - q$ lines for which the hyperoval is external to its line. Now choose a hyperoval for which its line is external. If $q > 2$ there is another line m external to the hyperoval, and so $l * m$ has weight 4 in the $\mathbb{Z}_2$ code which is less than $q + 2$, a contradiction.

It is not known if Type T2 pairings can occur in projective planes of even orders more than 2. However, we can show that both types of pairings occur in a projective plane of order 2, and that the corresponding $\mathbb{Z}_4$ code is $\mathbb{Z}_2$ linear.

Now we construct a code over $\mathbb{Z}_4$ from π when q is odd. Recall that $\mathcal{P}$ and $\mathcal{L}$ denote the sets of points and lines, respectively, of π and $|\mathcal{P} \cup \mathcal{L}| = 2(q^2 + q + 1)$. Let δ be a map from point P to the set of lines not through P and let ∂ be the dual mapping. δ extends to the linear coboundary map from a set of points S to lines that intersect S in opposite parity to $|S|$. Note that $|S| \equiv |\delta S| \pmod 2$. There is one-to-one correspondence between the boolean algebra of all subsets of $\mathcal{P} \cup \mathcal{L}$ and the vector space $\mathbb{F}_2^{2(q^2+q+1)}$[14]. For any two subsets E and F of $\mathcal{P} \cup \mathcal{L}$ the symmetric difference $E \triangle F$ corresponds to addition $E + F$ in $\mathbb{F}_2^{2(q^2+q+1)}$. The size of any subset F of $\mathcal{P} \cup \mathcal{L}$ corresponds to the weight of a vector F. With this correspondence in hand we can define the associated binary codes $\mathcal{C}_A$ and $\mathcal{C}_B$. Let $\mathcal{C}_A$ be the set of all even sets of points of $\mathcal{P}$ together with the sets of lines that intersect these points an odd number of times. It was shown in [14] that $\mathcal{C}_A$ is a binary linear doubly even self-orthogonal code with parameters $[2(q^2 + q + 1), q^2 + q, 2q + 2]$. Also $\mathcal{C}_A^\perp$ is a code with parameters $[2(q^2 + q + 1), q^2 + q + 2, q + 2]$. Another binary code $\mathcal{C}_B$ was defined as a union

of $\mathcal{C}_A$ with one of its cosets $\mathcal{C}_A \cup (\mathcal{C}_A + \mathcal{P} + \mathcal{L})$. It was shown that $\mathcal{C}_B$ is a binary linear self-dual code with parameters $[2(q^2+q+1), q^2+q+1, 2q]$. A main property of these codes is the following.

Proposition 1 *[14]* $\mathcal{C}_A \subset \mathcal{C}_B \subset \mathcal{C}_A^{\perp}$.

In view of this and Condition 2 of Theorem 1 we get the following.

Theorem 4 *Let π be a projective plane of odd order q. Then there exists a self-orthogonal linear code $\mathcal{C}_\pi(q)$ over $\mathbb{Z}_4$ with reduction code $\mathcal{C}_A$, torsion code $\mathcal{C}_B$, length $2(q^2+q+1)$, 2-dimension $2q^2+2q+1$, and minimum Hamming weight $2q$.*

The remainder of this section considers properties of the code $\mathcal{C}_\pi(q)$. A generator matrix of the code $\mathcal{C}_B$ is $G(\mathcal{C}_B) = [I_{q^2+q+1} \mid D]$, where D is the complement of the incidence matrix of the projective plane with i^{th} row d_i for $1 \le i \le q^2+q+1$, and I_{q^2+q+1} is the identity matrix of order q^2+q+1. Then by Theorem 4, the generator matrix of the code $\mathcal{C}_\pi(q)$ over $\mathbb{Z}_4$ is

$$\left[\begin{array}{cccc|c|c} 1 & 0 & \dots & 0 & 1 & d_1 + d_{q^2+q+1} \\ q+1 & 1 & \dots & 0 & 1 & d_2 + d_{q^2+q+1} \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ q+1 & q+1 & \dots & 1 & 1 & d_{q^2+q} + d_{q^2+q+1} \\ 0 & 0 & \dots & 0 & 2 & 2d_{q^2+q+1} \end{array}\right]. \tag{6}$$

This code is self-orthogonal by construction.

Example 1 *For $q = 1$, $\mathcal{C}_B$ is a $[6,3,2]$ code with generator matrix $G_B = [I_3|D]$ where D is generated by the cyclic shifts of the vector (100). The weight distribution is $A(0) = 1, A(2) = 3, A(4) = 3$, and $A(6) = 1$. The code $\mathcal{C}_A$ is a $[6,2,4]$ optimal code with weight distribution $A(0) = 1$, and $A(4) = 3$. The code $\mathcal{C}_\pi(1)$ is a $[6,5,2,4]$ Type α code with generator matrix*

$$\left[\begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 & 0 & 2 \end{array}\right].$$

The Hamming, Lee and Euclidean Weight distributions of this code are

i	$A_H(i)$	i	$A_L(i)$	i	$A_L(i)$
0	1	0	1	0	1
2	3	4	11	4	8
4	11	6	8	8	12
5	8	8	11	16	3
6	9	12	1	24	1

Example 2 *For $q = 3$, $\mathcal{C}_B$ is a $[26, 13, 6]$ code with generator matrix $G_B = [I_{13}|D]$ where D is generated by the cyclic shifts of the vector* (0010111110111). *The weight distribution is* $A(0) = 1, A(6) = 52, A(8) = 390, A(10) = 1313, A(12) = 2340, A(14) = 2340,$
$A(16) = 1313, A(18) = 390, A(20) = 52,$ *and* $A(26) = 1$. *The code $\mathcal{C}_A$ is a $[26, 12, 8]$ optimal code with weight distribution $A(0) = 1, A(8) = 390, A(12) = 2340, A(16) = 1313,$ and $A(20) = 52$.*
$\mathcal{C}_\pi(3)$ is a $[26, 25, 6, 8]$ code with generator matrix

$$\left[\begin{array}{cccccccccccc|c|cccccccccccc}
1&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&1&0&0&0&0&1&1&0&0&1\\
0&1&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&1&0&0&0&1&0&1&0&1\\
0&0&1&0&0&0&0&0&0&0&0&0&1&1&0&0&1&0&1&0&0&1&0&0&1&1\\
0&0&0&1&0&0&0&0&0&0&0&0&1&1&0&1&1&1&0&1&0&1&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&1&0&1&1&0&1&1&0&0&0&1\\
0&0&0&0&0&1&0&0&0&0&0&0&1&1&1&1&0&0&1&1&0&0&0&0&0&1\\
0&0&0&0&0&0&1&0&0&0&0&0&1&1&0&0&0&0&0&1&1&1&1&0&0&1\\
0&0&0&0&0&0&0&1&0&0&0&0&1&1&0&1&1&0&0&0&1&0&0&1&0&1\\
0&0&0&0&0&0&0&0&1&0&0&0&1&1&0&1&0&1&0&0&0&0&1&0&1&1\\
0&0&0&0&0&0&0&0&0&1&0&0&1&1&0&1&0&0&1&0&0&1&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&1&0&0&0&1&0&1&0&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&0&0&1&1&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&2&0&2&0&2&2&2&2&2&0&2&2&2&0
\end{array}\right].$$

The Hamming and Lee weight distributions are given by

i	$A_H(i)$	i	$A_L(i)$
0	1	0	1
6	52	8	312
8	702	12	3172
10	4433	14	29952
12	75660	16	94718
13	29952	18	868608
14	459420	20	1403753
15	868608	22	4722432
16	1085929	24	5477628
17	4642560	26	8353280
18	2009358	28	5477628
19	8087040	30	4722432
20	4868812	32	1403753
21	4722432	34	868608
22	4485000	36	94718
23	1134848	38	29952
24	948480	40	3172
25	109824	44	312
26	21321	52	1

.

We can modify the above construction to get an equivalent code over $\mathbb{Z}_4$ via a projective plane of odd order in a more natural way. Instead of (6), we take the generator matrix as

$$\left[\begin{array}{cccc|c|c} 1 & 0 & \dots & 0 & 3 & d_1 + 3d_{q^2+q+1} \\ q+1 & 1 & \dots & 0 & 3 & d_2 + 3d_{q^2+q+1} \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ q+1 & q+1 & \dots & 1 & 3 & d_{q^2+q} + 3d_{q^2+q+1} \\ 0 & 0 & \dots & 0 & 2 & 2d_{q^2+q+1} \end{array}\right]. \tag{7}$$

Thus we have the following.

Proposition 2 *The code generated by (7) is a self-orthogonal code over $\mathbb{Z}_4$ for any odd q. Further it can be made into a self-dual code, which we denote by $\mathcal{C}_\pi'$, of length $n = 2(q^2+q+1)$ by deleting the last row and adding the two rows $22\dots2|2|0\dots0$ and $00\dots0|0|2\dots2$.*

Proof. A combination of the first q^2+q rows in (7) is dependent to the last three rows and since the last two rows are independent to the top rows, we delete the last row to yield a self-dual code over $\mathbb{Z}_4$ of length $n = 2(q^2+q+1)$.

□

Remark 2 *For $q = 1$, $\mathcal{C}'_\pi$ is the unique self-dual code of length 6 over $\mathbb{Z}_4$ [24].*

If $q \equiv 3 \pmod 4$ then the above code can be defined more naturally in geometrical language from a projective plane π of odd order q as follows. Let P be a general point of π and δP denote its boundary. Then $\mathcal{C}_\pi(q)$ is set of all codewords $\sum_{P\in\pi} \alpha_P(P + \delta P)$ such that $\sum_{P\in\pi} \alpha_P \equiv 0 \pmod 4$. This can be made into a self-dual code by adding $\sum_{P\in\pi} 2P$ and $\sum_{L\in\pi} 2L$.

The next result determines the Shadow of the self-dual code of Proposition 2.

Proposition 3 *Let the Shadow of the self-dual code $\mathcal{C}'_\pi$ be S_q. If $\mathbf{c} \in \mathcal{C}'_\pi$ then either $\mathbf{c} \in S_q$ or $\mathbf{c}$ is not in S_q.*

Proof. A typical codeword in $\mathcal{C}'_\pi$ will have Euclidean weight $E = b + 4c + d$ if it has a components 0, b components 1, c components 2 and d components 3. Also due to the presence of the all 2 vector we also have Euclidean weight $E' = 4a+b+d$. But $a+b+c+d = 1 \pmod 4$ and $b+2c+3d = 0 \pmod 4$, by definition, hence $a-c$ is odd. As $E'-E = 4(a-c)$, we have the result.

□

The construction of the code $\mathcal{C}_\pi(q)$ can be generalized in the following way.

Lemma 6 *Let $G = [A|B]$ be the generator matrix of a self-orthogonal code over $\mathbb{Z}_4$ where the partition of G is compatible with a matrix X such that $XX^t = I$. Then the code generated by $G' = [A|BX]$ will also be self-orthogonal code over $\mathbb{Z}_4$.*

Proof. It is straightforward to check that the matrix G' satisfies the required property.
□

Remark 3 *In Lemma 6, if we substitute $G = (10\cdots01)(10\cdots03)$, where the parathensis denotes that all the cyclic shifts are taken, and $X = D$ is the complement of the incident matrix of the projective plane, then we obtain $\mathcal{C}_\pi(q)$. Note that*

$$DD^t = \begin{cases} 2J - I & if\ q \equiv 3 \pmod 4 \\ I & if\ q \equiv 1 \pmod 4 \end{cases}.$$

Similar results have been investigated recently in [11]. The next result shows that the code $\mathcal{C}_\pi(q)$ is not $\mathbb{Z}_2$-linear i.e, its image under the Gray map is a non-linear binary code.

Theorem 5 *The Gray image of $\mathcal{C}_\pi(q)$ is a $\left(4(q^2+q+1), 2^{2q^2+2q+1}, 2q+2\right)$ binary non-linear code.*

Proof. The result follows from Theorem 2. Let $\mathbf{u}$ and $\mathbf{v}$ be the first two rows of (6). If $\bar{}$ denotes reduction modulo 2 and $\star$ denotes component-wise multiplication, then $2\bar{\mathbf{u}} \star \bar{\mathbf{v}}$ is a vector of weight $q+1$. Since there is no vector of weight $q+1$ in $\mathcal{C}_B$ this vector does not belong to $\mathcal{C}_\pi(q)$.

□

4 Projective Planes and Quantum Codes

Any binary self-orthogonal code give rise to a binary quantum code via the CRSS construction [5]. The binary self-orthogonal codes from projective planes of odd order given in Section 3 will also provide a class of quantum codes. In this section, we list the parameters of such quantum codes. The necessary theorem is the following [5].

Theorem 6 *Let $\mathcal{C}$ be an $[n,k,d]$ binary self-orthogonal code with dual $\mathcal{C}^\perp$ having parameters $[n, n-k, d^\perp]$. Let $\left\{\mathbf{w_j} : 0 \le j \le 2^{n-2k}\right\}$ be a system of coset representatives of $\mathcal{C}^\perp \backslash \mathcal{C}$. Then the 2^{n-2k} mutually orthogonal states*

$$|\psi_j\rangle = \frac{1}{\sqrt{|\mathcal{C}|}} \sum_{\mathbf{c}\in\mathcal{C}} |\mathbf{c} + w_j\rangle,$$

span a binary quantum code with parameters $[[n, n-2k]]$ of length n and dimension 2^{n-2k} and minimum weight d', where

$$d' = \min\left\{wt(\mathbf{c}) \mid \mathbf{c} \in \mathcal{C}^\perp \setminus \mathcal{C}\right\} \ge d^\perp.$$

Applying the above theorem to code $\mathcal{C}_A$ will yield a $[[2(q^2+q+1), 2]]$ quantum code. This code encodes 2-qubits into $2(q^2+q+1)$ qubits.

If we apply the above theorem to the code $\mathcal{C}_B$ we obtain a quantum code with parameters $[[2(q^2+q+1), 0, 2q]]$. If we compare it with the existing Tables [5], one finds that for $q=3$ we get an optimal quantum code.

If $q \equiv 1 \pmod 4$, the binary self-dual code $\mathcal{C}_B$ can be extended to a unique doubly-even self-dual code $\mathcal{C}_D$ with parameters $[2(q^2+q+2), q^2+q+2, q+3]$ [14]. Applying the above theorem to $\mathcal{C}_D$ yields a binary quantum code with parameters $[[2(q^2+q+2), 0, q+3]]$.

If $q \equiv 3 \pmod 4$, the binary self-dual code $\mathcal{C}_B$ can be extended to a doubly-even self-dual code $\mathcal{C}_E$ with parameters $[2(q^2+q+4), q^2+q+4, 4]$ [14]. Applying the above theorem to $\mathcal{C}_E$ yields a binary quantum code with parameters $[[2(q^2+q+4), 0, 4]]$.

5 Self-Orthogonal Quasi-Twisted Codes

In this section, we present the results of a search for best self-orthogonal quasi-twisted codes. Because the Gray map connects the Lee weights of a $\mathbb{Z}_4$ code with the Hamming weights of a $\mathbb{Z}_2$ code, we cosider here only best Lee weight codes. This search employed a stochastic optimization algorithm, tabu search [12, 13, 19]. This method has been shown to produce optimal or near-optimal solutions to difficult optimization problems with a reasonable amount of computational effort. For an extensive survey of optimization methods in coding theory, with an emphasis on stochastic procedures, see [22].

Tabu search is based on local search, which means that starting from an arbitrary initial solution, a series of solutions is obtained so that every new solution only differs slightly from the previous one. A potential new solution is called a *neighbor* of the old solution, and all neighbors of a given solution constitute the *neighborhood* of that solution. To evaluate the quality of solutions, a *cost function* is needed. Tabu search always proceeds to a best possible solution in the neighborhood of the current solution.

To ensure that the search does not loop on a subset of solutions, recent solutions are stored in a so-called tabu list, and these are then not allowed for a certain period of time.

The search criterion used here was the minimum weight, and the cost function was chosen so as to maximize this weight, with the added condition that the resulting code be self-orthogonal. It was found that it is best to first find a code with a specified even minimum distance, then check for orthogonality.

Table 1 presents the minimum weights of the best codes obtained. Note that it was shown in [18] that self-dual QT codes exist only for lengths a multiple of 8 (m a multiple of 4). The first rows of the twistulant matrices of the QT codes listed in Table 1 are compiled in Tables 2 - 5. Since this is the first compiled table of $\mathbb{Z}_4$ codes (self-orthogonal or otherwise), it is not possible to compare these codes with previous results. However, using the Gray map, it is possible to compare these codes with the best binary linear codes [3] with even minimum distance. Of the 111 entries in Table 1, 54 or almost half attain the best known distance for the corresponding binary code. Hence the class of self-orthogonal QT codes contains many good codes.

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Table 1: Maximum Minimum Lee Distances for Best Self-Orthogonal (pm, m) QC Codes over Z_4

	p																
m	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
2	–	4	8	8	12	14	16	16	20	22	24	26	28	32	32	34	36
3	–	–	8	14	16	18	22	24	28	30	34	38	40	44	48	50	52
4	4	8	12	16	22	24	28	32	36	40	44	48	54	56	60	64	70
5	–	–	16	20	24	30	34	40	44	48	54	58	64	68	74	78	84
6	–	12	16	22	28	34	40	44	50	56	60	66	72	80	86	90	96
7	–	–	18	24	32	38	44	48	56	62	68	76	82	88	94	102	110
8	8	14	22	26	34	40	46	54	62	68	78	86	92	100	104	114	122

Table 2: First Rows of the Best Quaternary Self-Orthogonal QT Codes $m = 2, 3$

code	m	d	$b_i(x)$
(6,2)	2	4	1, 1, 13
(8,2)	2	8	1, 2, 12, 13
(10,2)	2	8	12, 2, 13, 2, 12
(12,2)	2	12	1, 13, 12, 12, 1, 11
(14,2)	2	14	12, 13, 1, 1, 12, 2, 11
(16,2)	2	16	1, 13, 1, 1, 12, 2, 12, 12
(18,2)	2	16	11, 1, 1, 1, 22, 13, 2, 22, 12
(20,2)	2	20	1, 1, 1, 12, 2, 12, 13, 12, 11, 11
(22,2)	2	22	1, 11, 13, 12, 12, 12, 1, 2, 2, 1, 11
(24,2)	2	24	11, 12, 1, 1, 1, 1, 12, 13, 22, 2, 2, 13
(26,2)	2	26	12, 1, 1, 1, 1, 11, 12, 12, 12, 11, 13, 2, 13
(28,2)	2	28	11, 1, 1, 12, 13, 13, 1, 22, 12, 12, 12, 12, 2, 11
(30,2)	2	30	12, 12, 12, 1, 1, 1, 1, 13, 13, 12, 2, 13, 2, 2, 11
(32,2)	2	32	11, 1, 1, 1, 1, 1, 11, 11, 13, 22, 12, 2, 2, 2, 12, 12
(34,2)	2	34	1, 1, 1, 1, 1, 12, 13, 11, 12, 11, 2, 12, 11, 12, 12, 2, 13
(36,2)	2	36	12, 12, 1, 1, 1, 1, 1, 1, 2, 12, 11, 11, 2, 13, 13, 11, 12, 22
(12,3)	3	8	1, 122, 21, 12
(15,3)	3	14	1, 122, 11, 12, 113
(18,3)	3	16	122, 123, 123, 113, 111, 122
(21,3)	3	18	1, 112, 123, 13, 122, 12, 111
(24,3)	3	22	1, 1, 113, 123, 112, 11, 112, 113
(27,3)	3	24	12, 111, 1, 1, 123, 112, 123, 11, 13
(30,3)	3	28	1, 1, 113, 11, 113, 122, 12, 11, 122, 12
(33,3)	3	30	12, 12, 12, 1, 11, 122, 11, 11, 113, 113, 111
(36,3)	3	34	1, 111, 123, 113, 1, 12, 122, 13, 12, 13, 11, 122
(39,3)	3	38	1, 123, 1, 11, 21, 123, 13, 12, 113, 111, 122, 2, 122
(42,3)	3	40	1, 21, 123, 12, 1, 12, 22, 21, 112, 2, 111, 11, 112, 113
(45,3)	3	44	122, 12, 21, 1, 1, 112, 11, 122, 11, 112, 113, 123, 2, 13, 113
(48,3)	3	48	112, 123, 112, 12, 1, 11, 21, 13, 111, 11, 21, 122, 12, 22, 113, 2
(51,3)	3	50	11, 12, 1, 12, 21, 112, 113, 12, 122, 13, 1, 122, 111, 113, 11, 21, 123
(54,3)	3	52	112, 1, 112, 111, 123, 113, 21, 21, 123, 12, 113, 12, 13, 11, 113, 122, 122, 122

Table 3: First Rows of the Best Quaternary Self-Orthogonal QT Codes $m = 4, 5$

code	m	d	$b_i(x)$
(8,4)	4	4	1, 133
(12,4)	4	8	1132, 123, 1213
(16,4)	4	12	112, 113, 102, 11
(20,4)	4	16	123, 122, 13, 11, 12
(24,4)	4	22	1132, 131, 122, 112, 1222, 1122
(28,4)	4	24	102, 111, 1, 12, 11, 1232, 1223
(32,4)	4	28	1223, 113, 121, 11, 133, 102, 1113, 1123
(36,4)	4	32	132, 11, 111, 133, 211, 122, 112, 202, 1222
(40,4)	4	36	122, 1113, 1, 1222, 211, 121, 111, 132, 113, 221
(44,4)	4	40	133, 212, 11, 1, 101, 112, 121, 123, 1112, 122, 221
(48,4)	4	44	211, 121, 1123, 122, 132, 113, 1222, 212, 131, 131, 123, 1
(52,4)	4	48	212, 12, 1232, 13, 131, 1212, 11, 221, 1132, 133, 1222, 1112, 22
(56,4)	4	54	113, 133, 1, 102, 131, 221, 1112, 1132, 122, 1213, 11, 21, 1222, 211
(60,4)	4	56	133, 102, 131, 1213, 1, 22, 1222, 21, 113, 132, 213, 1122, 211, 112, 1223
(64,4)	4	60	1113, 132, 1, 122, 111, 21, 133, 1222, 1112, 123, 102, 1, 12, 213, 1123, 1132
(68,4)	4	64	1113, 111, 12, 111, 2, 133, 1, 102, 123, 1223, 222, 1222, 121, 1132, 112, 132, 1223
(72,4)	4	70	1132, 1, 1222, 11, 1213, 221, 133, 122, 1112, 213, 212, 12, 1223, 1122, 1123, 211, 131, 112
(20,5)	5	16	1121, 2133, 11122, 1011
(25,5)	5	20	11, 133, 111, 1202, 11212
(30,5)	5	24	1113, 133, 131, 122, 1112, 11222
(35,5)	5	30	1021, 2221, 1321, 131, 1123, 11233, 1331
(40,5)	5	34	2212, 1323, 11112, 1223, 1122, 1232, 133, 12123
(45,5)	5	40	11312, 12223, 133, 1331, 1031, 1032, 1133, 11212, 2111
(50,5)	5	44	132, 11322, 1022, 1132, 1031, 1, 213, 1033, 2111, 11223
(55,5)	5	48	11113, 11112, 2021, 1202, 1132, 131, 1033, 122, 1322, 1031, 1102
(60,5)	5	54	11313, 11, 12, 2221, 101, 2021, 1211, 12122, 1221, 131, 111, 1123
(65,5)	5	58	2211, 1323, 211, 1132, 2122, 11322, 1311, 1123, 12223, 11223, 1202, 1212, 1332
(70,5)	5	64	1122, 1031, 111, 2132, 2112, 12223, 12222, 122, 21, 1111, 113, 12, 12313, 12123
(75,5)	5	68	2131, 132, 11, 1132, 2123, 201, 11223, 1011, 1033, 123, 11222, 12, 1131, 12122, 12213
(80,5)	5	74	11122, 1022, 12122, 1132, 1122, 12223, 11112, 12222, 12132, 1032, 1323, 11232, 2211, 1333, 11313, 12313
(85,5)	5	78	113, 201, 11223, 1031, 122, 1022, 1331, 21, 12222, 1302, 133, 11112, 11132, 11212, 1321, 2213, 2221
(90,5)	5	84	1211, 11313, 2221, 1132, 1221, 1323, 1, 2112, 11123, 12213, 1121, 211, 101, 12, 221, 131, 2212, 11112

Table 4: First Rows of the Best Quaternary Self-Orthogonal QT Codes $m = 6, 7$

code	m	d	$b_i(x)$
(18,6)	6	12	21, 111, 11321
(24,6)	6	16	21, 21313, 22213, 12
(30,6)	6	22	10323, 1333, 1, 102, 112122
(36,6)	6	28	1203, 121223, 112, 10321, 1, 13311
(42,6)	6	34	1131, 13333, 12333, 13022, 1323, 10112, 13302
(48,6)	6	40	20212, 12113, 1011, 1113, 21212, 1031, 13321, 1303
(54,6)	6	44	11212, 1301, 101, 2221, 20211, 11222, 11033, 1031, 11312
(60,6)	6	50	22113, 12, 13022, 2111, 10131, 21213, 121313, 13202, 10122, 1301
(66,6)	6	56	10213, 12022, 1223, 10121, 21133, 1031, 22211, 13111, 11013, 12211, 2013
(72,6)	6	60	111222, 11111, 1323, 12, 112322, 10121, 12322, 122232, 13, 112313, 22122, 13022
(78,6)	6	66	21113, 112223, 1, 201, 1101, 11322, 112323, 11031, 213, 21211, 1122, 121222, 13132
(84,6)	6	72	2, 1, 13, 112, 10213, 122232, 112233, 22111, 12012, 112133, 1131, 1213, 2103, 12121
(96,6)	6	86	2122, 2101, 2011, 10123, 10213, 112132, 21311, 12312, 21312, 121213, 1212, 113132, 122, 21221, 12313, 11131
(102,6)	6	90	213, 21313, 1, 10311, 12223, 122123, 21222, 21333, 22211, 10223, 1303, 13022, 10103, 1133, 11111, 2113, 133
(108,6)	6	96	12231, 11112, 2121, 133, 1, 113223, 21321, 12, 11131, 10211, 1332, 22113, 1002, 111332, 111313, 12312, 1303, 13221
(28,7)	7	18	12, 1123132, 13111, 103311
(35,7)	7	24	12122, 20111, 1, 1113111, 21033
(42,7)	7	32	1213, 2102, 113313, 111113, 210222, 10112
(49,7)	7	38	131132, 113032, 101312, 1231231, 1122122, 213133, 21212
(56,7)	7	44	211, 1212122, 113012, 210213, 1301, 1213212, 1132, 20123
(63,7)	7	48	102, 102212, 123231, 221123, 221222, 112213, 212113, 1123212, 123111
(70,7)	7	56	21112, 1, 113122, 1113131, 1011, 11212, 21012, 121131, 133211, 10121
(77,7)	7	62	21031, 10133, 20221, 12023, 22102, 1213, 101233, 1303, 133132, 11131, 102202
(84,7)	7	68	101021, 11, 122213, 202021, 1132312, 1131221, 130221, 11022, 1131321, 113132, 102133, 1022
(91,7)	7	76	10023, 213233, 13, 122202, 213211, 12022, 13122, 212231, 12103, 123212, 131133, 1123132, 202111
(98,7)	7	82	12132, 121312, 122313, 110131, 1113231, 1131221, 111213, 132212, 2132, 1132322, 112032, 1301, 121133, 1122211
(105,7)	7	88	13232, 211121, 131231, 13023, 1222312, 123222, 132113, 101201, 21103, 202112, 1112122, 110222, 222123, 12031, 103231
(112,7)	7	94	113, 1113, 10122, 123022, 112233, 111211, 13221, 21303, 102233, 10331 2, 1132132, 202222, 1111211, 12, 131133, 21
(119,7)	7	102	101333, 11223, 123, 13132, 112232, 10111, 120121, 1112222, 12211, 1213122, 13112, 1132131, 131323, 101322, 1111312, 132, 123233 321, 231311
(126,7)	7	110	103322, 11112, 111233, 2013, 101102, 1222, 1212221, 21132, 111012, 13022, 1323, 113133, 102012, 11231, 1232, 110122, 12201, 10133 12201, 20211

Table 5: First Rows of the Best Quaternary Self-Orthogonal QT Codes $m = 8$

code	m	d	$b_i(x)$
(16,8)	8	8	111, 1323123
(24,8)	8	14	12103, 11031, 213113
(32,8)	8	22	1121211, 1310231, 103021, 100213
(40,8)	8	26	12, 1, 2211212, 1313221, 11321312
(48,8)	8	34	221, 13333, 1201313, 1022311, 1323112, 210113
(56,8)	8	40	10231, 120132, 120301, 1102022, 110211, 1223222, 2021121
(64,8)	8	46	133, 1221023, 1111032, 202132, 213, 1311102, 1110122, 1223323
(72,8)	8	54	112, 101123, 110221, 1113122, 1111332, 1230232, 1311021, 13212, 1132323
(80,8)	8	62	121012, 1, 21312, 1033122, 221311, 11111212, 1133323, 11221221, 1012132, 1232331
(88,8)	8	68	1011, 11013, 1211023, 122031, 1131332, 1122231, 110233, 122022, 1103112, 121102, 12122212
(96,8)	8	78	132223, 1210223, 1012332, 130302, 10223, 12233, 103111, 1012333, 1023133, 201212, 1103232, 1312231
(104,8)	8	86	131111, 1212202, 102002, 212, 1302112, 123211, 132201, 1013322, 132032, 100231, 111311, 10112, 2021331
(112,8)	8	92	1020131, 12, 1122012, 2111213, 1133322, 2221332, 112302, 1222331, 1103121, 2211123, 202213, 132322, 1111333, 13212
(120,8)	8	100	10223, 11122131, 1221013, 11221312, 112102, 1113313, 2111, 1321112, 123332, 1321211, 12, 102303, 121333, 1101302, 210123
(128,8)	8	104	1233, 1213, 102002, 222221, 201232, 132301, 11322321, 1333321, 2123231, 1032312, 113212, 1330231, 2132221, 1332332 1112031, 1333322
(136,8)	8	114	1113102, 1012113, 1111311, 1101121, 2122211, 1203113, 1033131, 2111, 132032, 1212313, 1031323, 2113022, 211102, 130311 1223102, 1231031, 131231
(144,8)	8	122	2212131, 101232, 123, 110233, 12222221, 103012, 12203, 2102322, 103013, 1311232, 12222321, 1112023, 11232122, 1230221 11312311, 2131022, 113321, 1021132

Footnote

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