HW05 Solution by Zengshi Chen 11/09/04 ①

11.13. (a) $x[n] = u[n] + 3\delta(n-1) \leftrightarrow X(z) = \frac{z}{z-1} + 3z^{-1}$

$v[n] = u[n-2] \leftrightarrow V(z) = \frac{z}{z-1} z^{-2} = \frac{z^{-1}}{z-1}$

$x(n) * v(n) \leftrightarrow X(z)V(z) = Y(z)$

$\Rightarrow X(z)V(z) = \left(\frac{z}{z-1} + 3z^{-1}\right)\frac{z^{-1}}{z-1} = \frac{1}{(z-1)^2} + \frac{3z^{-2}}{z-1}$

$\Rightarrow X(z)V(z) = z^{-1}\frac{z}{(z-1)^2} + \frac{3z^{-3}z}{z-1}$

$\Leftrightarrow (n-1)u(n-1) + 3u(n-3)$

properties used $\delta(n-q) \leftrightarrow z^{-q}$, $q=1,2,\dots$

$\frac{z}{(z-1)^2} \leftrightarrow nu(n)$, $u(n) \leftrightarrow \frac{z}{z-1}$

$z^{-q}V(z) \leftrightarrow v(n-q)$, $q = 1, 2, 3, \dots$

Let's check the result by $n=4$.

$y(4) = \sum_{k=0}^{4} x(k)v(4-k) = x(0)v(4) + x(1)v(3) + x(2)v(2) + x(3)v(1) + x(4)v(0)$

$= 1\cdot1 + 4\cdot1 + 1\cdot1 + 0 + 0 = \underline{\underline{6}}$ equal

$y(4) = (4-1)u(4-1) + 3u(4-3) = 3+3 = \underline{\underline{6}}$

or by $n=2$, $y(2) = \sum_{k=0}^{2} x(k)v(2-k) = \underline{\underline{1}}$ equal

$y(2) = (2-1)u(2-1) + 3u(2-1) = \underline{\underline{1}}$

(b) $x(n) = u(n) \leftrightarrow X(z) = \frac{z}{z-1}$

$v(n) = nu(n) \leftrightarrow \frac{z}{(z-1)^2} = V(z)$

$Y(z) = X(z)V(z) = \frac{z^2}{(z-1)^3} \leftrightarrow y(n) = \frac{1}{2}n(n+1)u(n)$

Property used: $n(n+1)a^n u(n) \leftrightarrow \frac{2az^2}{(z-a)^3}$

(c) $x(n) = \sin(\frac{\pi n}{2}) u(n) \leftrightarrow \frac{z}{z^2+1} = X(z)$

$v(n) = e^{-n} u(n-2) \leftrightarrow v(n) = e^{-2} e^{-(n-2)} u(n-2)$

$\leftrightarrow V(z) = e^{-2} z^{-2} \frac{z}{z-e^{-1}} = \frac{e^{-2}}{z(z-e^{-1})}$

$Y(z) = X(z)V(z) = \frac{z}{z^2+1} \frac{e^{-2}}{z(z-e^{-1})} = \frac{e^{-2}}{(z^2+1)(z-e^{-1})}$

$= \frac{a}{z-e^{-1}} + \frac{bz+c}{z^2+1}$

$\frac{az^2+a+bz^2+cz-be^{-1}z-ce^{-1}}{(z-e^{-1})(z^2+1)}$

$= \frac{(a+b)z^2+(c-be^{-1})z+a-ce^{-1}}{(z-e^{-1})(z^2+1)}$

$\Rightarrow a+b=0, \quad c-be^{-1}=0, \quad a-ce^{-1}=e^{-2}$

$\Rightarrow a = \frac{e^{-2}}{1+e^{-2}}, \quad b=-a, \quad c = be^{-1} = -ae^{-1}$

$\Rightarrow Y(z) = \frac{a}{z-e^{-1}} + \frac{-az-e^{-1}a}{z^2+1}$

$\Rightarrow Y(z) = a\left(\frac{1}{z-e^{-1}} - \frac{z}{z^2+1} - e^{-1}\frac{1}{z^2+1}\right)$

$\Rightarrow Y(z) = az^{-1}\left(\frac{z}{z-e^{-1}} - \frac{z^2}{z^2+1} - e^{-1}\frac{z}{z^2+1}\right)$

$\Rightarrow y(n) = a\left((e^{-1})^{(n-1)}u(n-1) - \cos\frac{\pi}{2}(n-1)\,u(n-1) - e^{-1}\sin\frac{\pi}{2}(n-1)\,u(n-1)\right)$

$\Rightarrow y(n) = \frac{e^{-2}}{1+e^{-2}}\left(e^{-(n-1)} - \sin\frac{\pi}{2}n + e^{-1}\cos\frac{n\pi}{2}\right)u(n-1)$

Properties used: $\frac{z}{z-a} \leftrightarrow a^n u(n)$, $\cos\frac{\pi}{2}n\, u(n) \leftrightarrow \frac{z^2}{z^2+1}$, $\sin\frac{\pi}{2}n\, u(n) \leftrightarrow \frac{z}{z^2+1}$

$z^{-q} Q(z) \leftrightarrow Q(n-q), \quad q = 0, 1, 2, \cdots$

11.19. (a) $y(n+2) + y(n) = 2x(n+1) - x(n)$

$\Leftrightarrow$ $y(n) + y(n-2) = 2x(n-1) - x(n-2)$

$\Leftrightarrow$ $Y(z) + z^{-2}Y(z) = 2z^{-1}X(z) - z^{-2}X(z)$

$\Leftrightarrow$ $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{2z^{-1} - z^{-2}}{1 + z^{-2}} = \dfrac{2z-1}{z^2+1}$

$\Leftrightarrow$ $H(z) = 2\dfrac{z}{z^2+1} - \dfrac{z}{z^2+1}z^{-1}$

$\Leftrightarrow$ $\underline{h(n) = 2\sin\frac{\pi}{2}n\, u(n) - \sin\frac{\pi}{2}(n-1)\, u(n-1)}$

(b) $U(z) = \dfrac{z}{z-1} \leftrightarrow u(n) = 1,\ n \geq 0$

$Y(z) = H(z)U(z) = \dfrac{(2z-1)z}{(z^2+1)(z-1)} = z\left(\dfrac{2z-1}{(z^2+1)(z-1)}\right)$

$\Rightarrow$ $Y(z) = z\left(\dfrac{a}{z-1} + \dfrac{bz+c}{z^2+1}\right)$

$\Rightarrow$ $Y(z) = z\,\dfrac{az^2 + a + bz^2 + cz - bz - c}{(z-1)(z^2+1)}$

$= z \cdot \dfrac{(a+b)z^2 + (c-b)z + a - c}{(z-1)(z^2+1)}$

$\Rightarrow$ $a+b=0,\quad c-b=2,\quad a-c=-1$

$\Rightarrow$ $a = \frac{1}{2},\quad b = -\frac{1}{2},\quad c = \frac{3}{2}$

$\Rightarrow$ $Y(z) = z\left(\frac{1}{2}\frac{1}{z-1} - \frac{1}{2}\frac{z}{z^2+1} + \frac{3}{2}\frac{1}{z^2+1}\right)$

$\Rightarrow$ $Y(z) = \frac{1}{2}\frac{z}{z-1} - \frac{1}{2}\frac{z^2}{z^2+1} + \frac{3}{2}\frac{z}{z^2+1}$

11.19. (a) $y(n+2) + y(n) = 2x(n+1) - x(n)$

$$\Leftrightarrow\ y(n) + y(n-2) = 2x(n-1) - x(n-2)$$

$$\Leftrightarrow\ Y(z) + z^{-2}Y(z) = 2z^{-1}X(z) - z^{-2}X(z)$$

$$\Leftrightarrow\ H(z) = \frac{Y(z)}{X(z)} = \frac{2z^{-1} - z^{-2}}{1+z^{-2}} = \frac{2z-1}{z^2+1}$$

$$\Leftrightarrow\ H(z) = 2\frac{z}{z^2+1} - \frac{z}{z^2+1}z^{-1}$$

$$\Leftrightarrow\ \underline{\underline{h(n) = 2\sin\tfrac{\pi}{2}n\ u(n) - \sin\tfrac{\pi}{2}(n-1)\ u(n-1)}}$$

(b) $U(z) = \frac{z}{z-1} \Leftrightarrow u(n) = 1,\ n \geq 0$

$$Y(z) = H(z)\,U(z) = \frac{(2z-1)z}{(z^2+1)(z-1)} = z\left(\frac{2z-1}{(z^2+1)(z-1)}\right)$$

$$\Rightarrow\ Y(z) = z\left(\frac{a}{z-1} + \frac{bz+c}{z^2+1}\right)$$

$$\Rightarrow\ Y(z) = z\,\frac{az^2 + a + bz^2 + cz - bz - c}{(z-1)(z^2+1)}$$

$$= z\cdot\frac{(a+b)z^2 + (c-b)z + a - c}{(z-1)(z^2+1)}$$

$$\Rightarrow\ a+b=0,\quad c-b=2,\quad a-c=-1$$

$$\Rightarrow\ a = \tfrac{1}{2},\quad b = -\tfrac{1}{2},\quad c = \tfrac{3}{2}$$

$$\Rightarrow\ Y(z) = z\left(\frac{1}{2}\frac{1}{z-1} - \frac{1}{2}\frac{z}{z^2+1} + \frac{3}{2}\frac{1}{z^2+1}\right)$$

$$\Rightarrow\ Y(z) = \frac{1}{2}\frac{z}{z-1} - \frac{1}{2}\frac{z^2}{z^2+1} + \frac{3}{2}\frac{z}{z^2+1}$$

$$y(n) = \frac{1}{2}u(n) - \frac{1}{2}\cos\frac{\pi}{2}n\, u(n) + \frac{3}{2}\sin\frac{\pi}{2}n\, u(n)$$

(c) $y(n+2) + y(n) = 2x(n+1) - x(n)$

$\rightarrow y(n) + y(n-2) = 2x(n-1) - x(n-2)$

$\rightarrow Y(z) + z^{-2}Y(z) + y(-2) + z^{-1}y(-1)$

$= 2z^{-1}X(z) + 2x(-1) - z^{-2}X(z) - x(-2) - z^{-1}x(-1)$

$y(-2) = 2,\ y(-1) = 3,\ x(-2) = 0,\ x(-1) = 0$

$\rightarrow (1+z^{-2})Y(z) + 2 + 3z^{-1} = (2z^{-1} - z^{-2})X(z)$

$\rightarrow Y(z) = \dfrac{2z^{-1} - z^{-2}}{1+z^{-2}}X(z) - \dfrac{2+3z^{-1}}{1+z^{-2}}$

$\because\ x(n) = 2^n u(n)$

$\Rightarrow X(z) = \dfrac{z}{z-2}$

$\Rightarrow Y(z) = \dfrac{2z-1}{z^2+1}\dfrac{z}{z-2} - \dfrac{2z^2+3z}{z^2+1}$

$Y(z) = \dfrac{(2z-1)z - (2z^2+3z)(z-2)}{(z^2+1)(z-2)}$

$Y(z) = \dfrac{-2z^3+3z^2+5z}{(z^2+1)(z-2)} = z\dfrac{-2z^2+3z+5}{(z^2+1)(z-2)}$

$\Rightarrow Y(z) = z\cdot\left(\dfrac{a}{z-2} + \dfrac{bz+c}{z^2+1}\right) = z\dfrac{az^2+a+bz^2+cz-2bz-2c}{(z-2)(z^2+1)}$

$\Rightarrow Y(z) = \dfrac{(a+b)z^2 + (c-2b)z + a - 2c}{(z-2)(z^2+1)}z$

$a+b = -2,\ c-2b = 3,\ a-2c = 5$

$a = \frac{3}{5},\ b = -\frac{13}{5},\ c = -\frac{11}{5}$

$$Y(z) = z\left(\frac{3}{5}\frac{1}{z-2} - \frac{13}{5}\frac{z}{z^2+1} - \frac{11}{5}\frac{1}{z^2+1}\right)$$

$$\Rightarrow Y(z) = \frac{3}{5}\frac{z}{z-2} - \frac{13}{5}\frac{z^2}{z^2+1} - \frac{11}{5}\frac{z}{z^2+1}$$

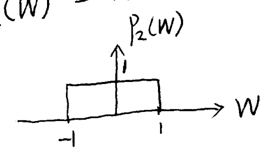
$$\Rightarrow y(n) = \left(\frac{3}{5}2^n - \frac{13}{5}\cos\frac{\pi}{2}n - \frac{11}{5}\sin\frac{\pi n}{2}\right)u(n)$$

12.1 (a) $x(t) = 3\,\mathrm{sinc}^2\left(\frac{t}{2\pi}\right)$

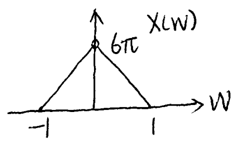
(Table 4.2)
function $P_\tau(t)$ on Page 11 of text

$$x(t) = 3\,\frac{2}{2}\,\mathrm{sinc}^2\left(\frac{2t}{4\pi}\right)$$

$$\Rightarrow \quad x(t) \leftrightarrow 3\cdot 2\pi\left(1 - \frac{2|w|}{2}\right)P_2(w) = X(w)$$

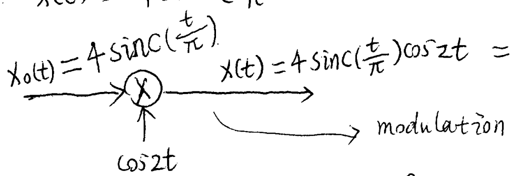


$$P_2(w) = \begin{cases} 1, & -\frac{2}{2} \le w < \frac{2}{2} \\ 0, & w < -\frac{2}{2},\ w \geq \frac{2}{2} \end{cases}$$



From the spectrum of the signal, the highest frequency is $w_h = 1$ rad/sec
By Sampling theorem, the sampling frequency must be $w_s \geq 2w_h = 2$ rad/sec.

(b) $x(t) = 4\,\mathrm{sinc}\left(\frac{t}{\pi}\right)\cos 2t$



$$X(w) = \int_{-\infty}^{\infty} x(t)e^{-jwt}\,dt = \int_{-\infty}^{\infty} x_0(t)\cos 2t\, e^{-jwt}\,dt$$

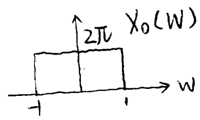
$$= \int_{-\infty}^{\infty} x_0(t)\frac{e^{j2t}+e^{-j2t}}{2}e^{-jwt}\,dt = \frac{1}{2}\int_{-\infty}^{\infty} x_0(t)e^{-j(w-2)t}\,dt + \frac{1}{2}\int_{-\infty}^{\infty} x_0(t)e^{-j(w+2)t}\,dt$$

$X(W) = \frac{1}{2} X_0(W-2) + \frac{1}{2} X_0(W+2)$ (page 252, text).

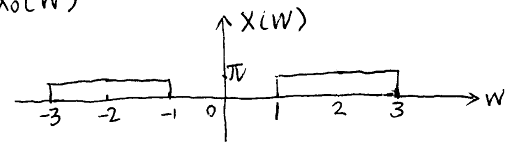
By table 4.2,

$$x_0(t) = 4\,\mathrm{sinc}\left(\frac{t}{\pi}\right) = 2\cdot 2\,\mathrm{sinc}\left(\frac{2t}{2\pi}\right)$$

$$\leftrightarrow 2\pi P_2(W) = X_0(W)$$



$\Rightarrow$



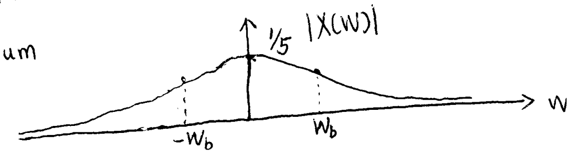
Apparently, $W_h = 3$, $W_s \geq 2W_h = 6$ rad/sec

cc) $x(t) = e^{-5t} u(t)$

$$x(t) = e^{-5t} u(t) \leftrightarrow X(W) = \frac{1}{jW+5}$$

$$\Rightarrow |X(W)| = \frac{1}{\sqrt{W^2+5^2}}$$

frequency spectrum



This is not bandlimited spectrum, so you cannot find the highest frequency. Usually, we cut off the spectrum outside the bandwidth and keep the remaing. We approximately take the bandwidth as the highest frequency. let's define the bandwidth in this way,

$$\frac{|X(W_b)|}{|X(0)|} = \frac{1}{\sqrt{2}} \Rightarrow \frac{\frac{1}{\sqrt{W_b^2+5^2}}}{\frac{1}{\sqrt{0^2+5^2}}} = \frac{1}{\sqrt{2}} \Rightarrow W_b = 5 \text{ rad/sec}$$

$\Rightarrow W_s \geq 2W_b = 10$ rad/sec. You can choose $W_s = 20 W_b$ to get a reasonable small distortion due to aliasing.